GEOMETRIC AND ELECTROMAGNETIC RESPONSE AND ANOMALIES IN TOPOLOGICAL PHASES OF MATTER

BY

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DISSERTATION

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Abstract

In this thesis we characterize various topological phases of matter by studying their response to external perturbations. In the first half of the thesis we study incompressible fluid phases with odd, or Hall, viscosity. This kind of viscosity was originally discovered in quantum Hall fluids, where it can be computed by studying the stress response of the quantum Hall fluid to time-dependent area-preserving deformations, which are a particular example of a *geometric* perturbation. In Chapter 2 we study classical two-dimensional fluids with Hall viscosity in their own right. In particular, we study the physics of a swimmer immersed in such a fluid, and in the low Reynolds number regime in which the effects of conventional viscous forces outweigh the effects of inertial forces. There we find that the Hall viscosity leads to a number of striking effects on the swimmer's motion, for example the swimmer can rotate itself only by changing its area. In Chapter 3 we study Hall viscosity directly in the context of the quantum Hall effect. There we compute the Hall viscosity in the Chern-Simons matrix model of the Laughlin states, which is a certain regularization of the noncommutative Chern-Simons theory of these states proposed by Susskind. Our calculations show that these noncommutative theories are able to describe the most important contribution, namely the *guiding center* contribution, to the Hall viscosity (and other geometric response properties) of the Laughlin states.

In the second half of the thesis we study *electromagnetic* response and anomalies in two families of bosonic symmetry-protected topological phases. These are the bosonic integer quantum Hall (BIQH) and bosonic topological insulator (BTI) phases. Although these phases were originally defined in three and four spacetime dimensions, respectively, we generalize them to all higher spacetime dimensions (odd dimensions for BIQH and even for BTI). We then study the bulk electromagnetic response of these theories as well as the anomalies in quantum field theories which can describe the boundaries of these phases. Our results include the discovery of an interesting quantization of the response coefficients (analogous to Hall conductance) for these phases, which depends explicitly on the spacetime dimension. We also study perturbative and global anomalies in the boundary theories for the BIQH and BTI phases, and we prove that the anomalies we compute are robust to a large set of deformations of the boundary theories which preserve the symmetry of the BIQH and BTI phases. We provide an introduction to perturbative and global anomalies in Chapter 1 of the thesis so that readers can follow the discussion of anomalies for the BIQH and BTI states in Chapter 5.

To the memory of Professor Christopher L. Henley

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Chapter 1

Introduction

1.1 Basic aspects of topological phases of matter

The study of topological phases of matter is now one of the most active areas of research in condensed matter physics. In fact, interest in the peculiar properties of these phases has spread beyond condensed matter physics, as witnessed by the fact that several well-known high-energy physicists and mathematicians are now working on the topic. There are several reasons for this intense interest. From an experimental point of view these phases are of interest because (i) they display precise quantization of physical properties (for example Hall conductance), and (ii) these properties are robust in the sense that they do not change even when the material is subjected to various kinds of perturbations, doped with impurities, etc. These properties may make some topological phases of matter extremely useful for the construction of new devices and the development of new technologies. For example, Kitaev's proposal [1] to use topological phases with anyons to perform fault-tolerant quantum computation has been a major source of motivation for much of the work on topological phases in the past two decades. From a theoretical point of view, topological phases are of interest because of their connection to deep ideas in physics and mathematics such as anomalies in quantum field theory and the topology of fiber bundles. In addition, the challenge of developing a theoretical understanding of topological phases has forced theorists to move beyond traditional ways of understanding phases of matter. For example, the existence of distinct phases of matter which cannot be distinguished by their symmetry properties has forced condensed matter physicists to move beyond Landau's idea [2] that different phases can be distinguished by the symmetries that they break.

Let us describe some basic properties that all topological phases have in common. All topological phases have an energy gap to excitations in the bulk of the system. That is, the Hamiltonian describing the topological phase will have a finite number of nearly degenerate low energy states, or *ground states*, and then a large energy gap separating these states from the rest of the spectrum. In addition, the gap in the spectrum stays finite even as the size of the system is taken to infinity. The large gap in the spectrum is the reason why the properties of a topological phase are robust to perturbations. Typically, the interesting properties of a topological phase are unaffected by perturbations which do not close the gap in the bulk spectrum. Two examples of "interesting properties" that are protected by the bulk energy gap are gapless conducting modes at the boundary of the phase and quantized transport properties such as Hall conductance in the bulk of a phase.

In discussing topological phases, we must distinguish between two kinds of such phases. The first kind of topological phase is the kind which is said to possess *intrinsic topological order* [3]. Hallmarks of intrinsic topological order are as follows. First, the system can have bulk excitations with *fractional* quantum numbers such as charge and statistics. For example, the $\nu = \frac{1}{m}$ Laughlin fractional quantum Hall state, which occurs (for m odd) in systems whose basic building blocks are electrons of charge $-e$, has "quasihole" excitations with a fractional charge $\frac{e}{m}$ and fractional exchange statistics of $e^{i\frac{\pi}{m}}$. By fractional exchange statistics we mean that $e^{i\frac{\pi}{m}}$ is the phase picked up by the wave function of the system after an (adiabatic) exchange of the positions of two quasihole excitations. The fractional charge and statistics of the quasihole excitations is quite striking considering the fact that these excitations arise in a system constructed solely from electrons with charge $-e$ and fermionic exchange statistics. We note here that particles with fractional statistics are called *anyons* [4]. There are two kinds of anyons which can exist in phases with intrinsic topological order. The first kind are called Abelian anyons and are characterized by the fact that the wave function of the system picks up a phase (which is not equal to ± 1) when two such anyons are exchanged or braided. The second kind are called non-Abelian anyons. Under an exchange of two non-Abelian anyons the state vector for the system is multiplied by a unitary *operator* (as opposed to a simple phase), and this operator rotates the state vector within a subspace of several degenerate states which form a basis for the sector of the Hilbert space that describes the topological phase with several non-Abelian anyon excitations present.

The second property which characterizes systems with intrinsic topological order is topology-dependent *ground state degeneracy*. Hamiltonians which describe a phase with intrinsic topological order will have a ground state degeneracy which depends on the topology of the spatial manifold that the topological phase is placed on. For our example of the $\nu = \frac{1}{m}$ Laughlin state, a Hamiltonian which describes this state is predicted to have a ground state degeneracy of m^g when the system is placed on a Riemann surface (a closed, orientable, two-dimensional manifold) of genus g (the genus g counts the number of "holes" in the manifold). In realistic systems the exact degeneracy of these multiple ground states is expected to be split by an amount which is exponentially small in the system size, so that the exact degeneracy is recovered in the thermodynamic limit of an infinitely large system. This topology-dependent ground state degeneracy can also be used as a diagnostic to test whether two different Hamiltonians describe the same topological phase.

Phases with intrinsic topological order are *long-range entangled*, meaning that the quantum states (or wave functions) describing these phases exhibit entanglement over macroscopically large length scales (for example length scales of the order of the system size). In fact, the topology-dependent ground state degeneracy of these systems is often associated with the presence of loop or string structures in the wave function, in which the loops or strings wrap around the different "handles" or *cycles* of a spatial manifold with nontrivial topology. This kind of extended structure in the wave functions for phases with intrinsic topological order explains why these phases are able to "sense" global properties of the spatial manifold that they live on, even when the Hamiltonian which describes the phase only has local interactions.

The second kind of topological phase is the kind which *does not* possess intrinsic topological order. These systems do not have quasiparticles with fractional quantum numbers, and Hamiltonians which describe such phases always have a single unique ground state on any closed spatial manifold, regardless of the topology of the manifold. One might then ask what, if any, are the special properties of this second kind of topological phase. The answer is that while these phases are trivial in the bulk, they typically display interesting properties at their boundary (or at an interface with a different phase). For example, in one spatial dimension there exist topological wires made up of electrons which bind a fractional charge $-\frac{e}{2}$ at their boundary [5]. Higher-dimensional examples include the celebrated quantum spin Hall system in two dimensions [6] and the time-reversal invariant topological insulator in three dimensions [7, 8]. The quantum states which describe these phases typically exhibit entanglement only for distances on the order of the correlation length (perhaps the length of a few unit cells), and so these phases are said to be *short-range entangled*.

The special properties of this second kind of topological phase are not as robust as the first kind, and it is generally the case that an additional symmetry is required to ensure the protection of this second kind of topological phase. For the examples mentioned above, the quantization of the boundary charge $-\frac{e}{2}$ in the one-dimensional topological wire is protected by inversion or charge-conjugation symmetry, and the quantum spin Hall and topological insulator phases are both protected by time-reversal symmetry. There are, however, certain topological phases which do not possess intrinsic topological order and also do not require any symmetry for their protection. Two examples in two spatial dimensions are the integer quantum Hall state of electrons and the "E8" state of bosons [9]. These systems do not require any symmetries for their protection because both systems possess completely *chiral* edge states when they are placed on a spatial manifold with a boundary. The chirality of the edge states is enough to protect these systems because a completely chiral edge cannot be gapped out by back-scattering perturbations (there is no channel moving in the opposite direction for a particle to be scattered into).

The idea that a topological phase requires a certain symmetry for its protection leads directly to the concept of *symmetry-protected topological* (SPT) phases (see, for example, Refs. [10–15]). SPT phases are topological phases of the second kind which are protected (i.e., robust to perturbations, etc.) only when a certain symmetry is enforced. This symmetry is described mathematically by the choice of a group G, for example we would have $G = U(1)$ for a SPT phase of charged bosons which only requires $U(1)$ charge conservation symmetry for its protection. Any Hamiltonian which describes a SPT phase must possess the symmetry of the group G which protects that phase, and the unique ground state of this Hamiltonian must not break the symmetry spontaneously (i.e., the ground state must transform trivially under the action of G).

The interesting properties of SPT phases, including their response to perturbations such as electromagnetic fields, can often be understood in terms of the anomalous properties of the boundaries of these phases. Some examples of such properties are as follows. In one spatial dimension the boundary of the system can transform in a *projective* representation of the group G, which is a representation of G in which the group multiplication law is obeyed only up to a phase. This can be thought of as an example of the fractionalization of a symmetry group, similar to the concepts of fractional charge and statistics that we discussed above. A simple example of this phenomenon is the topological phase of spin chains with spins of magnitude $s = 1$ on each site. This topological phase is protected by the symmetry of the group $G = SO(3)$ (spins with $s = 1$ transform in the fundamental representation of $SO(3)$), but in this topological phase a lone spin $s = \frac{1}{2}$ degree of freedom is localized at each boundary of the chain. The boundary spin one-half transforms in the fundamental representation of $SU(2)$, which is a projective representation of $SO(3)$. This topological phase of spin one chains is often called the "Haldane phase" because of Haldane's prediction of a gap in spin one Heisenberg chains [16]. A concrete example of a Hamiltonian with a ground state in the Haldane phase is the Affleck-Kennedy-Lieb-Tasaki (AKLT) Hamiltonian [17].

In two spatial dimensions the boundary of a SPT phase may possess gapless edge states. However, these states can usually be gapped out if the symmetry of the SPT phase is broken, and so the edge states require the symmetry of the group G for their protection. A simple example of this phenomenon occurs in the bosonic integer quantum Hall (BIQH) state [9, 18], which is a SPT phase of charged bosons which is protected by $G = U(1)$ charge conservation symmetry. The edge of this system features two counter-propagating modes, but only one of these modes carries charge. Therefore, the charge transport at the edge is chiral, but the energy transport is not. If the $U(1)$ symmetry is preserved then the edge is protected because any backscattering term that one might add at the edge must scatter between the charged edge mode and the neutral edge mode, and such a term necessarily breaks the $U(1)$ symmetry. Therefore, the gapless edge of the BIQH state is only protected if the $U(1)$ symmetry of this state is preserved.

In three spatial dimensions the set of possibilities for the behavior of the boundary of a SPT phase is even richer than in one or two spatial dimensions. Here we only mention one possibility which is unique to three spatial dimensions. In three spatial dimensions it is possible for a SPT phase to have a gapped boundary which retains the symmetry of the group G, but in this case the boundary theory must also possess intrinsic topological order. In this situation the boundary theory will possess anyon excitations, and these anyons will transform in nontrivial ways under the action of the symmetry group G of the SPT phase. A simple example occurs at the boundary of the bosonic topological insulator (BTI) phase [19, 20], where the boundary theory has intrinsic topological order and possesses Abelian anyons which transform under the symmetries of the BTI phase (which are $U(1)$ charge conservation and $\mathbb{Z}_2^T$ time-reversal symmetry).

In the preceding paragraphs we used the vague phrase "anomalous properties" to describe the interesting features which distinguish the boundary of a SPT phase from the boundary of a trivial phase. Our use of the word "anomalous" was not an accident. It turns out that the boundary of a SPT phase typically exhibits an *anomaly* in the sense in which the word is used in quantum field theory. More precisely, the boundary of a SPT phase exhibits what is known as a *'t Hooft anomaly* [21] for the symmetry group G which protects the SPT phase. Roughly speaking, this means that the boundary theory cannot be consistently coupled to a background gauge field for the group G. Since the consideration of anomalies is central to the results presented in Chapters 4 and 5 of the thesis, we give a short introduction to 't Hooft anomalies (including two examples) in the last section of this Chapter.

If we take a broad view of the field of topological phases of matter, then we can see that all theoretical studies of these phases can be sorted into two main categories. The first category is *classification*. Theoretical works in this category attempt to enumerate all possible topologically distinct phases, perhaps subject to some restrictions. For example one can ask how many SPT phases there are in two spatial dimensions which are made up of bosons and possess only a $U(1)$ symmetry (which could represent a physical charge conservation symmetry). The BIQH phase that we discussed above is an example of such a SPT phase. The second category is *characterization*. Theoretical works in this category focus on understanding the physical properties which distinguish a particular topological phase from other topological phases, from trivial phases, or from other distinct phases such as gapless phases. It is clear that both categories of study are important. Indeed, a long-term goal of the entire field could be a complete classification of all phases as well as a complete characterization of these phases which would allow one to distinguish different phases by their physical properties. Our running example of the BIQH phase also serves as a nice example in which both of these goals have already been realized. The distinct SPT phases of bosons in two spatial dimensions and with $U(1)$ symmetry are labeled by an integer $k \in \mathbb{Z}$, and these phases can be distinguished physically by their Hall conductance which is given in terms of k by $\sigma_H = 2k\frac{q^2}{h}$, where q is the charge of the bosons [9].

The work represented in this thesis falls into the second category discussed above, i.e., we focus on the characterization of particular topological phases. Our main tool to characterize these phases is to study their response to external perturbations, and in the thesis we study the response of topological phases to two different kinds of perturbations. In Chapters 2 and 3 we consider *geometric* perturbations, which involve deforming the system in some way by stretching or straining it, perhaps in a time-dependent manner. In particular, in topological phases in which the electrons can be understood as forming an incompressible fluid (for example in the fractional quantum Hall effect), one can calculate a viscosity which is defined as the response of the system to a time-dependent strain [22]. In Chapters 4 and 5 we consider *electromagnetic* perturbations. Specifically, we study the response of SPT phases of charged bosons to externally applied electromagnetic fields. As we mentioned above, the electromagnetic response of the bulk of a SPT phase is intimately related to anomalies in quantum field theories which describe the boundary of these phases, and

Chapter 5 contains a detailed study of anomalies in the boundary theories for these SPT phases of charged bosons. In the next section we give a detailed summary of the work which is contained in this thesis.

1.2 Overview of the thesis

We now give an overview of each Chapter of the thesis. As we mentioned above, Chapters 2 and 3 of the thesis are related to the response of topological phases to geometric perturbations. Both of these Chapters deal with the study of fluid systems with *odd* viscosity, also known as *Hall* viscosity, which is a particular kind of geometric response. In what follows we use the terms odd and Hall viscosity interchangeably. In our overview of Chapter 2 of the thesis we use the term odd viscosity, as Chapter 2 concerns this kind of viscosity in generic two-dimensional fluids. In our overview of Chapter 3 we use the term Hall viscosity, because in that Chapter we study this viscosity in the specific context of fractional quantum Hall systems.

To define the odd viscosity first recall that in a fluid the viscosity tensor η_{ijkl} is the coefficient which determines the linear response of the *stress* tensor T_{ij} to the *rate of strain* tensor v_{kl} via the relation

$$T_{ij} = \eta_{ijkl} v_{kl} \ , \tag{1.2.1}$$

where i, j, k, l are spatial indices and we sum over repeated indices in this expression (here we adopt the notation of Chapter 2 of the thesis). The rate of strain tensor v_{kl} can be expressed in terms of the components v_k of the fluid velocity as $v_{kl} = \frac{1}{2}(\partial_k v_l + \partial_l v_k)$, and v_{kl} is also equal to the time derivative of the strain tensor (for small deformations) that one usually studies in continuum mechanics. The viscosity tensor η_{ijkl} is symmetric under the exchanges $i \leftrightarrow j$ and $k \leftrightarrow l$, and this is because the stress tensor T_{ij} and rate of strain tensor v_{kl} are symmetric under these exchanges. However, it is possible for η_{ijkl} to have contributions which are either even or odd under the exchange of the first two indices $\{ij\}$ with the second two indices $\{kl\}$ (i.e., we send $\eta_{ijkl} \to \eta_{klij}$). Conventional viscosity (which causes dissipation in a fluid) is even under this kind of exchange of indices, while the odd viscosity is odd under this exchange. In two spatial dimensions an odd contribution to the viscosity tensor is compatible with rotational symmetry, and so it is possible for isotropic two-dimensional fluids to exhibit odd viscosity (see Chapter 2 for more details).

The odd viscosity was originally studied in the context of the integer quantum Hall effect [22], and this is why this viscosity is often called "Hall viscosity". The electrons in a system in the integer quantum Hall phase (and many fractional quantum Hall phases as well) form an incompressible fluid, and the Hall viscosity in these systems can be calculated by studying the linear response of the stress of this fluid to time-dependent area-preserving deformations (or strains) of the fluid. More generally, an odd contribution to the viscosity tensor can be present in any two-dimensional fluid system with broken time-reversal symmetry [23], and the unique properties of such fluids make them interesting objects of study in their own right.

The odd viscosity leads to a number of nonintuitive effects. For example, a rotating disk embedded in a two-dimensional fluid with ordinary viscosity experiences a torque which opposes the direction of rotation. However, in a fluid with odd viscosity the rotating disk would feel a radial *pressure* instead of a torque, and this pressure would squeeze or pull on the disk depending on the direction of the rotation. The fact that the effect of odd viscosity is different for the two directions that the disk could rotate in (clockwise vs. counterclockwise) is a reflection of the fact that odd viscosity only occurs in fluids with broken time-reversal symmetry. We discuss this example in more detail in Chapter 2 (see also Fig. 2.1 in that Chapter). Another interesting property of odd viscosity is that unlike ordinary viscosity in fluids, the odd viscosity is not associated with any dissipation of energy. Another way to state this fact is to say that the forces which arise from the odd viscosity contribution to the stress tensor do not do any net work on the fluid, and so there is no energy loss in the fluid due to the odd viscosity.

In Chapter 2 we study the physics of a swimmer immersed in a classical fluid possessing both conventional viscosity as well as odd viscosity, with the goal of understanding how the presence of odd viscosity affects the motion of the swimmer. We focus on the case of swimming at low Reynolds number, which is the regime in which (conventional) viscous effects dominate the fluid flow. The theory of swimming in fluids at low Reynolds number is a fascinating subfield of fluid dynamics [24]. Due to the special properties of this regime, in particular the fact that any motion is quickly damped out by conventional viscous forces, the theory of swimming at low Reynolds number admits a *geometric* formulation in which the motion of the swimmer is completely determined by the sequence of shapes assumed by the swimmer as it performs its swimming stroke. This geometric theory of the problem of swimming at low Reynolds number was developed by Shapere and Wilczek [25]. A notable feature of this geometric formulation of the swimming problem is the fact that it takes the form of a classical non-Abelian gauge theory. As a consequence, calculations in this geometric theory of swimming have a formal resemblance to calculations in Yang-Mills theory and in the theory of non-Abelian Berry phase in quantum mechanics.

In Chapter 2 we apply Shapere and Wilczek's geometric theory of swimming at low Reynolds number to the study of nearly circular swimmers in two-dimensional fluids with a non-vanishing odd viscosity (we also assume that the fluid has a conventional viscosity which is large enough so that the fluid is in the low Reynolds number regime). The odd viscosity gives an off-diagonal contribution to the fluid stress-tensor, and this results in a number of striking effects on the swimmer's motion. In particular, we find that a swimmer whose area is changing will experience a torque proportional to the rate of change of the area, with the constant of proportionality given by the coefficient η^o of odd viscosity. In Chapter 2 we derive a general theory of swimming in fluids with odd viscosity for a class of simple swimmers, and then we give a number of example swimming strokes which clearly demonstrate the differences between swimming in a fluid with conventional viscosity and a fluid which also has odd viscosity. We also include a discussion of the extension of the famous "Scallop theorem" of low Reynolds number swimming to the case where the fluid has a non-zero odd viscosity. A number of more technical results, including a proof of the torque-area relation for swimmers of more general shape, are explained in Appendix A. Chapter 2 is based on Ref. [26].

In Chapter 3 we again study Hall viscosity, but this time we study it directly in the context of the fractional quantum Hall effect. Specifically, we investigate the Hall viscosity in certain models of the Laughlin fractional quantum Hall states which model these states as charged fluids flowing on a *noncommutative* space. A noncommutative space is a space in which the coordinates do not commute with each other, and so the notion of a point in space becomes fuzzy. The simplest example of such a space is the noncommutative version of the plane $\mathbb{R}^2$ in which the "coordinates" $\hat{x}^a$, $a = 1, 2$, are actually operators (which is why we used a "hat" in the notation) which obey the commutation relation

$$[\hat{x}^1, \hat{x}^2] = i\theta \ . \tag{1.2.2}$$

Here θ is a real number with units of length squared, and this number sets the strength of the "noncommutativity" of the noncommutative plane. On this space there is an uncertainty principle which states that there is a smallest unit of area, proportional to θ, beyond which one cannot resolve the location of any object. In other words, the best one can do is to say that an object lies somewhere inside a box of size proportional to θ. There is a Hilbert space $\mathcal{H}_F$ associated with the algebra in Eq. (1.2.2) (i.e., this space provides a representation of the algebra of the noncommutative coordinates), and for many purposes one can regard $\mathcal{H}_F$ as the noncommutative analogue of the plane $\mathbb{R}^2$ (see Chapter 3 for more details).

In Ref. [27] Susskind proposed to model the Laughlin fractional quantum Hall states using Chern-Simons theory on the noncommutative plane. This proposal should be viewed as an extension of the more familiar description of the Laughlin states using $U(1)$ Abelian Chern-Simons theory (see, for example, Ch. 7 of Ref. [28]). Susskind arrived at this theory by first considering a model which describes a fractional quantum Hall state as a charged fluid in a constant magnetic field. He then notes that this model possesses a large symmetry, namely the symmetry of area-preserving diffeomorphisms of the reference coordinates of the fluid, and he further argues that this symmetry should not be present in a fluid made up of discrete objects such as electrons. He therefore proposes to deform this model by placing it on the noncommutative plane, because on that plane there is smallest area scale (proportional to θ) which one can think of as the area occupied by a single electron. The model that one gets after performing this noncommutative deformation of the fluid model turns out to be exactly the $U(1)$ Chern-Simons theory on the noncommutative plane. Therefore, a simple physical picture of the Chern-Simons theory on the noncommutative plane is that it is a model of a charged fluid in a constant magnetic field in which the fluid is made up of indivisible particles of area $\sim \theta$.

The noncommutative Chern-Simons theory captures many basic aspects of the Laughlin states, for example it gives the correct mean density of the electrons in the Laughlin states and the correct Hall conductance. It is therefore natural to ask whether this noncommutative model can also describe more complicated properties of the Laughlin states such as the Hall viscosity and other properties related to geometric response. The ordinary Chern-Simons description of the Laughlin states (which contains no dimensionful parameters) describes all of the topological properties of these states such as the ground state degeneracy on a Riemann surface and the braiding statistics of quasiparticle excitations. The noncommutative Chern-Simons theory differs from the ordinary Chern-Simons theory in that it contains the extra scale θ, and so it is natural to expect that the noncommutative theory can describe more than just the topological properties of the Laughlin states.

In Chapter 3 we study Hall viscosity and other aspects of geometric response in the Laughlin fractional quantum Hall states using a model which is closely related to the noncommutative Chern-Simons theory. The model that we actually study is a matrix quantum mechanics model known as the Chern-Simons matrix model (CSMM), and it was proposed by Polychronakos in Ref. [29] as a regularization of the noncommutative Chern-Simons theory description of the Laughlin states proposed earlier by Susskind. The CSMM is similar to the noncommutative Chern-Simons theory in the sense that it also describes a charged fluid on the noncommutative plane. However, in the case of the CSMM the fluid forms a droplet of finite area instead of occupying the entire noncommutative plane. This also means that the CSMM describes a quantum Hall droplet made up of a finite number of electrons, whereas Susskind's noncommutative Chern-Simons theory should be thought of as describing a system with an infinite number of electrons occupying the full noncommutative plane.

In our work in Chapter 3 we revisit the CSMM in light of recent developments on geometric response in the fractional quantum Hall effect, with the goal of determining whether the CSMM captures this aspect of the physics of the Laughlin states. We compute the Hall viscosity, Hall conductance in a non-uniform electric field, and the Hall viscosity in the presence of anisotropy (or "intrinsic geometry") for the Laughlin states using the CSMM description of these states. Our calculations show that the CSMM captures the guiding center contribution to the known values of these quantities in the Laughlin states, but lacks the Landau orbit contribution. The interesting correlations in a Laughlin state are contained entirely in the guiding center part of the state/wave function, and so we conclude that the CSMM accurately describes the most important aspects of the physics of the Laughlin states, including the Hall viscosity and other geometric properties of these states which are of current interest. Supplementary material for Chapter 3 is given in Appendix B. Chapter 3 is based on a recent arXiv preprint [30].

Chapters 4 and 5 of the thesis deal with electromagnetic response and anomalies in SPT phases of charged bosons. In these Chapters we focus our attention on two specific SPT phases of charged bosons, which are the BIQH and BTI phases mentioned in the previous section. We also define and study generalizations of the BIQH and BTI states to higher spacetime dimensions, and the study of these higher-dimensional phases represents a major portion of these Chapters of the thesis. The BIQH phase was originally studied in three spacetime dimensions [9, 18], but in Chapters 4 and5 of the thesis we study generalizations of this phase to all *odd* spacetime dimensions. The BTI phase was originally studied in four spacetime dimensions [19, 20], and in Chapters 4 and5 of the thesis we study generalizations of this phase to all *even* spacetime dimensions. Our main goal in this work was to understand the physical properties that distinguish these higher-dimensional BIQH and BTI phases from higher-dimensional analogues of the integer quantum Hall and topological insulator phases of free fermions.

Of all the possible SPT phases of bosons, the BIQH and BTI phases are particularly interesting because they are the closest analogue, in an interacting bosonic system, of the familiar integer quantum Hall and topological insulator phases of free fermions. At this point it is important to emphasize that interactions are absolutely crucial for the stabilization of any kind of topological phase in a system of bosons. This is because in the absence of interactions the ground state of a many-body system of bosons is a trivial condensate in which all bosons simultaneously occupy the single-body eigenstate with the lowest energy. Therefore, in the study of bosonic SPT phases one is forced to deal with interacting models (e.g., lattice models, quantum field theories, etc.) and this can make the study of topological phases of bosons extremely challenging. In Chapters 4 and 5 we study these phases using interacting bosonic quantum field theories, and in order to make progress in our study of these phases we use some fairly sophisticated techniques from the quantum field theory literature which are rarely used in condensed matter physics. In particular, we use the theory of *gauged Wess-Zumino actions* and the *equivariant localization* technique, which are both related to the mathematical subject of *equivariant cohomology*.

The necessity of interactions for topological phases of bosons should be contrasted with the situation for SPT phases of fermions. Since fermions obey the Pauli exclusion principle, they can form a topological phase even in the absence of interactions. Indeed, in many examples the topological properties of fermionic SPT phases are encoded in the twisting of Bloch wave functions across the Brillouin zone, which is a property of *single particles* occupying an electronic band in a solid. This is not to say that all fermionic SPT phases have a description in terms of single-particle physics. There are actually several examples of SPT phases of fermions which *require* interactions for their existence, for example the "interaction-enabled" topological crystalline phases that we studied in collaboration with Jeffrey C.Y. Teo [31].

The BIQH and BTI phases are both SPT phases of charged bosons, and so both phases require $U(1)$ charge conservation symmetry for their protection. The BTI state also requires an additional discrete symmetry which can be either anti-unitary time-reversal symmetry $\mathbb{Z}_2^T$ or unitary charge-conjugation symmetry $\mathbb{Z}_2^C$ depending on the spacetime dimension. The details of this issue are discussed in Chapter 4. Here we only mention that in both cases the total symmetry group for the BTI phase has the form $G = U(1) \rtimes \mathbb{Z}_2$, where the $\mathbb{Z}_2$ factor is either time-reversal or charge-conjugation and the semi-direct product "$\rtimes$" indicates that the action of the $U(1)$ and $\mathbb{Z}_2$ parts of G do not commute with each other. This fact about the additional $\mathbb{Z}_2$ symmetry is important because, as we describe in Chapter 4, one interesting phase which can occur at the boundary of the BTI is a boundary quantum Hall phase which spontaneously breaks the $\mathbb{Z}_2$ symmetry of the BTI phase.

In Chapter 4 we calculate the topological part of the electromagnetic response of BIQH phases in odd spacetime dimensions and BTI phases in even spacetime dimensions. We also define and study the topological part of the electromagnetic response of a putative gapless phase of bosons in even spacetime dimensions, which we refer to as a bosonic chiral semi-metal (BCSM) phase. This BCSM phase is designed to exhibit an electromagnetic response which has the same form as the response predicted for Weyl semi-metals [32, 33]. Therefore, our BCSM model should be thought of as a bosonic analogue of a Weyl semi-metal[1]. To compute these responses we combine two theoretical tools: the Nonlinear Sigma Model (NLSM) description of bosonic SPT phases [35, 36] and the method of gauged Wess-Zumino actions [37–41]. In the next few paragraphs we give a brief introduction to each of these tools.

The NLSM description of bosonic SPT phases is a generalization (to other contexts and to higher dimensions) of Haldane's description of an antiferromagnetic Heisenberg spin chain using an $O(3)$ NLSM [16]. Recall that the field in the $O(3)$ NLSM field theory is a three-component unit vector $\mathbf{n}(x,t)$ which is a function of the spatial coordinate x and the time t. For the antiferromagnetic spin s chain, which is the case of interest here, the field $\mathbf{n}(x,t)$ represents a coarse-grained version of the staggered spin $\frac{(-1)^j}{s}\mathbf{S}_j$, where $\mathbf{S}_j$ is the spin on site j of the spin chain (x and j are related as $x = ja_0$ where a_0 is the lattice spacing in the spin chain). In the antiferromagnetic case is it reasonable to expect that this staggered field takes on a smooth configuration when viewed at long distances.

The usual action for the $O(3)$ NLSM (or any NLSM for that matter) takes the form

$$S_0[\mathbf{n}] = \frac{1}{2g}\int dtdx\ \eta^{\mu\nu}(\partial_\mu n^a)(\partial_\nu n_a)\ , \tag{1.2.3}$$

where g is a coupling constant, $\mu = x,t$ is a spacetime index, $a = 1,2,3$ labels the three components of the NLSM field, and $\eta^{\mu\nu}$ is the Minkowski metric[2]. We also sum on all repeated indices in this expression. The action $S_0[\mathbf{n}]$ gives an energy cost when the field $\mathbf{n}(x,t)$ is not constant in spacetime, and so classically this action favors a uniform configuration of $\mathbf{n}(x,t)$, which would imply a state with antiferromagnetic order (recall that the NLSM field represents the *staggered* spin). However, in $1+1$ spacetime dimensions it is known that quantum effects destroy this ordering, and the quantum ground state of the $O(3)$ NLSM with action $S_0[\mathbf{n}]$ possesses the full $SO(3)$ symmetry of the NLSM (the quantum ground state transforms in the trivial representation of $SO(3)$). In addition, excitations over this ground state

[1] Our work here on a bosonic analogue of a Weyl semi-metal is a continuation of our previous work with Gil Young Cho on a bosonic analogue of a Dirac semi-metal in $2+1$ dimensions [34].

[2] We use a "mostly minus" convention so that as a matrix $\eta = \text{diag}(1,-1)$ in $1+1$ dimensions and $\eta = \text{diag}(1,-1,\dots,-1)$ in higher dimensions.

are gapped, and the lowest energy excitation transforms in the fundamental representation of $SO(3)$ and should be thought of as a spin wave. These properties of the $O(3)$ NLSM in $1+1$ dimensions were first derived in Refs. [42–45].

The $O(3)$ NLSM description of a spin s chain includes an additional term in the action besides the usual term $S_0[\mathbf{n}]$. This additional term is topological and is known as the *theta term*. It takes the form

$$S_\theta[\mathbf{n}] = \frac{\theta}{8\pi}\int dtdx\ \epsilon^{\mu\nu}\epsilon^{abc} n_a\partial_\mu n_b\partial_\nu n_c\ , \tag{1.2.4}$$

where again $\mu, \nu = t, x$ and $a, b, c = 1, 2, 3$. For the application to the spin s Heisenberg chain the coefficient of the theta term is quantized as $\theta = 2\pi s$ [46]. In particular, the spin $s = 1$ chain has $\theta = 2\pi$ and is predicted to be in a gapped phase (unlike the spin $s = \frac{1}{2}$ Heisenberg chain which is known to be gapless from its exact solution via the Bethe Ansatz [47]). More generally, the NLSM with theta term is expected to describe a gapped phase when θ is an integer multiple of 2π and a gapless phase when θ is an odd integer multiple of π. It is actually possible to prove that the energy spectrum of the theory is 2π periodic in the value of θ using a certain unitary operator in the quantized NLSM [48]. It then follows that the NLSM with $\theta = 2\pi k$ for any integer k must describe a gapped phase since the $k = 0$ case describes a gapped phase.

The theta term is topological because it does not require a metric $g_{\mu\nu}$ on spacetime to contract indices. Instead, the spacetime indices μ and ν are contracted using the Levi-Civita symbol $\epsilon^{\mu\nu}$, which is possible because we are in two spacetime dimensions and the theta term features two derivatives. To understand the physical meaning of the theta term it is useful to define the notion of the *target space* of the NLSM. The NLSM field can be viewed as a map from spacetime into some other space, and that other space is called the target space of the NLSM. In the case of the $O(3)$ NLSM the target space is just the unit two-sphere S^2, and this follows from the fact that a three-component unit vector defines a point on the unit two-sphere embedded in $\mathbb{R}^3$. The physical meaning of the theta term is that it counts the number of times that spacetime is "wrapped" around the target space of the NLSM (S^2 in this case). We make this interpretation more precise in Chapters 4 and 5 where we write the theta term in a more geometric way using the *volume form* on the target space of the NLSM. It is important to note here that the spacetime and the target space S^2 are both two-dimensional. This is not an accident. In fact, one can only construct a theta term for a NLSM if the target space of the NLSM and the spacetime have the same dimension. We comment on this more in the next paragraph.

In the NLSM description of bosonic SPT phases, phases in $D+1$ spacetime dimensions are described using an $O(D+2)$ NLSM with theta term. In the $O(D+2)$ NLSM the field is a $(D+2)$-component unit vector $\mathbf{n}(\mathbf{x}, t)$ which is a function of the D-dimensional spatial coordinate $\mathbf{x}$ and the time t. A $(D+2)$-component unit vector describes a point on the unit $(D+1)$-sphere S^{D+1} embedded in $\mathbb{R}^{D+2}$, and so the target space of this NLSM is $(D+1)$-dimensional. This is the same as the dimension of the spacetime, and so $D+2$ is the correct choice of the number of components for the NLSM if we want to incorporate a theta term into the action. Indeed, in the theta term the spacetime indices will now be contracted with the Levi-Civita symbol $\epsilon^{\mu_1\cdots\mu_{D+1}}$ and so the theta term must feature $D+1$ derivatives in order to contract all of these spacetime indices. If we compare with the structure of the theta term for the $O(3)$ NLSM, then we see that this requires the NLSM field to have one more component than the number of spacetime directions, which is $D+2$ in $D+1$ spacetime dimensions.

In the NLSM description a nontrivial bosonic SPT phase is described by a NLSM with theta term and with a coefficient $\theta = 2\pi k$ for some integer k. The reason for taking k to be an integer is that in this case the NLSM with theta term describes a gapped phase[3], just like the case of the antiferromagnetic Heisenberg chain for integer s in $D = 1$. The SPT phase is protected by the symmetry of a group G, and this information is encoded in the NLSM description by specifying an action of the group on the NLSM field. In other words, one needs a rule which assigns a particular rotation or reflection of the NLSM field for each element of G, and this rule should respect the group multiplication law. Mathematically, this rule is equivalent to a *group representation* $\sigma : G \to O(D+2)$, in which each element g of the group is mapped to a $(D+2)\times(D+2)$ orthogonal matrix $\sigma(g) \in O(D+2)$, where $O(D+2)$ is the group of orthogonal matrices of size $(D+2)\times(D+2)$. This group (or really the special orthogonal group $SO(D+2)$) is also the global symmetry group of the NLSM with theta term, and so embedding G inside this group guarantees that the NLSM description of the SPT phase actually respects the symmetry of that phase. We refer the reader to Chapter 4 for more details on how the symmetry of the SPT phase is encoded in the NLSM description.

As we discussed above, in the NLSM description the bulk of a SPT phase is modeled using a NLSM with theta term and with a coefficient $\theta = 2\pi k$, $k \in \mathbb{Z}$. There is also a way to model the boundary of a SPT phase within this formalism. In the NLSM description the boundary of the SPT phase is modeled using the same NLSM but with a *Wess-Zumino* term [49] instead of a theta term. The coefficient of the Wess-Zumino term is a certain integer, known as the *level* of the Wess-Zumino term, and in the NLSM description of a SPT phase the level of the Wess-Zumino term on the boundary is the same integer k that appears in the coefficient $\theta = 2\pi k$ of the theta term that appears in the action for the bulk of the SPT phase. Wess-Zumino terms will be familiar to many readers from the Lagrangians of the Wess-Zumino-Witten conformal field theories (see, for example, Ch. 15 of Ref. [50]), but they also appear in other contexts and in spacetime dimensions greater than two. We describe the detailed properties of Wess-Zumino terms in Chapters 4 and 5, and so here we only mention the main property of these terms which makes them useful for our analysis of the electromagnetic response of SPT phases.

In the context of the NLSM description of SPT phases one would like to couple the NLSM to a background gauge field for the symmetry group G of the SPT phase. It turns out that properly coupling the Wess-Zumino term to this

[3]More precisely, for $D > 1$ the NLSM with $\theta = 2\pi k$ for integer k only describes a gapped phase in the strong coupling limit $g \to \infty$. For weak coupling the theory spontaneously breaks the $O(D+2)$ symmetry, and so the weak coupling ground state of the NLSM does not represent a SPT phase.

background gauge field is quite subtle, and studying the gauged Wess-Zumino term in detail can provide important information about the 't Hooft anomaly of the boundary of the SPT phase, as well as the topological part of the response (to the background gauge field) of the bulk of the SPT phase. The problem which arises when we try to couple the Wess-Zumino term to a background gauge field can be traced back to the fact that the construction of the Wess-Zumino term involves an extension of the NLSM field configuration into an auxiliary, or fictitious, dimension of space. In terms of the NLSM field we have $\mathbf{n}(\mathbf{x},t) \to \tilde{\mathbf{n}}(\mathbf{x},t,s)$ where $\tilde{\mathbf{n}}(\mathbf{x},t,s)$ is the extension of the NLSM field into the extra dimension of space and s is a coordinate for this extra dimension (usually one chooses $s \in [0,1]$). The main difficulty in gauging the Wess-Zumino term stems from the physical requirement that the gauged action should describe physics in the original spacetime dimension, and not depend on the auxiliary dimension. In particular, this means that the usual minimal coupling procedure is not sufficient to gauge a Wess-Zumino term, since the minimally coupled Wess-Zumino term yields equations of motion which describe propagation of the NLSM field (couped to the background gauge field) in the auxiliary dimension of space. We refer the reader to Chapter 4 for a more detailed description of the difficulties involved in gauging the Wess-Zumino term.

In Chapter 4 we use the NLSM description of bosonic SPT phases and the theory of gauged Wess-Zumino actions to derive several surprising results about the electromagnetic responses of the BIQH, BTI, and BCSM phases. For BIQH states in spacetime dimension $2m-1$ ($m = 1, 2, \dots$), we find that the bulk response to an electromagnetic field A_μ is characterized by a Chern-Simons term for A_μ with a level quantized in integer multiples of $m!$ ("!" denotes a factorial). This result is interesting because it shows that in $2m-1$ spacetime dimensions the response of the BIQH phase to an applied electromagnetic field is $m!$ times larger than the response of the integer quantum Hall state of free fermions in the same dimension (generalizations of the integer quantum Hall state of fermions to higher spacetime dimensions have been studied in detail in Refs. [51–54]). We also show that BTI states in $2m$ spacetime dimensions can exhibit a $\mathbb{Z}_2$ breaking quantum Hall effect on their boundaries (recall that the BTI states also have an extra $\mathbb{Z}_2$ symmetry), with this boundary quantum Hall effect described by a Chern-Simons term at level $\frac{m!}{2}$. Here it is important to note that this boundary quantum Hall response is exactly half of what we found for the BIQH state that can exist intrinsically in $2m-1$ spacetime dimensions. In addition, this boundary quantum Hall response is $m!$ times larger than that of the topological insulator phase of free fermions in the same dimension (so the response of the BTI differs from that of its fermionic analogue by the same numerical factor that we found in the BIQH case). We also note here that our results for general spacetime dimensions agree with the known results for the BIQH state in $2+1$ dimensions (take $m = 2$ in the formula $2m-1$ for the spacetime dimension) and the BTI state in $3+1$ dimensions (take $m = 2$ in the formula $2m$ for the spacetime dimension), so this is encouraging.

The numerical factor of $m!$ which appears in our calculations for the BIQH and BTI states is rather mysterious, and it would be nice to understand it from some other point of view. For example it would be useful to have a mathematical or physical argument which explains why the electromagnetic response of the BIQH and BTI states should be quantized in this way. In Chapter 4 we provide a mathematical explanation for this quantization using a *gauge invariance* argument. Specifically, we show that the exponential of the Chern-Simons action for the external electromagnetic field A_μ on a general (closed) Euclidean spacetime manifold is only invariant under large $U(1)$ gauge transformations if the coefficient of the action is quantized to be an integer multiple of $m!$. This result provides a solid mathematical interpretation of the number $m!$ appearing in our response formulas. However, one thing which we were not able to do was to provide a solid *physical* explanation for this number, for example an argument based on the requirement of triviality of braiding statistics like the argument presented in Ref. [18] for the BIQH state in $2+1$ dimensions.

In Chapter 4 we also use this more general gauge invariance argument to characterize the electromagnetic *and* gravitational responses of fermionic SPT phases with $U(1)$ symmetry in all odd spacetime dimensions. Our analysis there is also closely related to the *Atiyah-Singer index theorem* [55] for the Dirac operator on a curved manifold and coupled to an electromagnetic field (physicists can find a very useful description of this theorem in Ref. [56]). We then use our gauged boundary actions for the BIQH and BTI states to (i) construct a bosonic analogue of a chiral semi-metal (the BCSM phase we discussed above) in even spacetime dimensions, (ii) show that the boundary of the BTI state exhibits a bosonic analogue of the parity anomaly of Dirac fermions in odd dimensions, and (iii) study anomaly inflow at domain walls on the boundary of BTI states. Several important formulas and additional results are explained and/or derived in the supplementary material in Appendix C. In particular, the supplementary material includes a discussion of the connection between *equivariant cohomology* and gauged Wess-Zumino actions, which allows us to give a mathematical interpretation of the gauged Wess-Zumino actions for the BIQH and BTI boundaries that we derive in Chapter 4. For example, our construction of the gauged Wess-Zumino action for the BTI boundary is equivalent to the construction of an *equivariant extension* of the volume form on even-dimensional spheres S^{2p} with respect to the action of the group $U(1)$ on these manifolds. Chapter 4 is based on Ref. [57] which we wrote in collaboration with Chao-Ming Jian and Peng Ye.

Finally, in Chapter 5 we study perturbative and global anomalies at the boundaries of the BIQH and BTI phases which were the main subject of Chapter 4 of the thesis. To study these anomalies we again rely on a description of the boundaries of these phases in terms of a NLSM with Wess-Zumino term. One of the main results of Chapter 5 is that these anomalies are robust against arbitrary smooth deformations of the target space of the NLSM which describes the phase, provided that the deformations also respect the symmetry of the phase. This result also implies that the topological part of the bulk electromagnetic response of these phases, which we computed in Chapter 4 using the boundary anomalies, is also robust to these deformations. The stability of these results to *symmetry-preserving* deformations of the model is exactly what one expects physically for any model of a SPT phase. However, it is highly nontrivial that this stability property can actually be proven for this class of models.

In the first part of Chapter 5 we discuss the perturbative $U(1)$ anomaly at the boundary of BIQH states in all odd spacetime dimensions. In the second part we study global anomalies at the boundary of BTI states in even spacetime dimensions. In Chapter 4 we argued that the boundary of the BTI phase exhibits a global anomaly which is an analogue of the *parity* anomaly of Dirac fermions in three spacetime dimensions [58–60]. In Chapter 5 we elevate this argument to a proof for the boundary of the $(1+1)$-dimensional BTI state by explicitly computing the partition function of the gauged NLSM describing the boundary. We then use the powerful *equivariant localization* technique to show that this global anomaly is robust to all smooth deformations of the target space of the NLSM which preserve the $U(1) \rtimes \mathbb{Z}_2$ symmetry of the BTI state. We also comment on the difficulties of generalizing this latter proof to higher dimensions. Finally, we discuss the expected low energy behavior of the boundary theories of the BIQH and BTI phases when their coupling constants are allowed to flow under the renormalization group. Supplementary material for Chapter 5, including an introduction to the equivariant localization technique (which allows for the exact evaluation of certain path integrals in systems with a hidden supersymmetry), is presented in Appendix D. Chapter 5 is based on Ref. [61].

1.3 Perturbative and global anomalies

In the last section of this Chapter we provide the reader with an introduction to the concepts of perturbative and global anomalies in quantum field theory. As we mentioned above, the study of anomalies is one of the best ways to characterize the boundary of a SPT phase. In addition, anomalies play a major role in the work that we present in Chapters 4 and 5 of the thesis, so it is important that we give enough background on anomalies for the reader to follow those Chapters. Simply put, a quantum field theory has an anomaly if a symmetry of the original classical theory is broken during the quantization procedure. This breaking of symmetry occurs because a quantum field theory must be regularized to be well-defined, and in some cases the method used to regularize the theory breaks a symmetry of the classical theory. If this happens then the theory is said to be anomalous. The first example of an anomaly in quantum field theory was the *axial anomaly* discovered by Adler, and by Bell and Jackiw, in the context of the calculation of certain divergent Feynman diagrams ("triangle" diagrams) in Refs. [62, 63]. This anomaly was later given a path integral interpretation by Fujikawa in Ref. [64]. These first examples are all examples of what is now known as a *perturbative* anomaly (see the definition below). The first example of a *global* anomaly (again, see below for the definition) was discovered by Witten in Ref. [65] in a theory of a doublet of Weyl fermions coupled to an $SU(2)$ gauge field.

In our examples below we study quantum field theories which have a certain global symmetry, and the anomaly only appears when we try to couple the theory to a background gauge field for this global symmetry. One then finds that even though the classical action for the theory coupled to the background gauge field is gauge invariant, the partition function for the quantum field theory coupled to the background gauge field is not. This type of anomaly is called a 't Hooft anomaly [21]. These 't Hooft anomalies are important physically because they occur at the boundary of SPT phases and turn out to be one of the best ways of characterizing SPT phases [66–70]. This is because these anomalies give a precise mathematical characterization of what is special about the boundary of a SPT phase relative to that of a trivial phase.

We discuss one example each of perturbative and global 't Hooft anomalies in this section. Our example of a theory with a perturbative anomaly is a single chiral Dirac fermion in $1 + 1$ dimensions. At the classical level this theory has a global $U(1)$ symmetry representing charge conservation, however, this theory has a perturbative 't Hooft anomaly when it is coupled to a background gauge field A_μ for the $U(1)$ symmetry. Physically, the anomaly implies that charge is not conserved in the chiral Dirac fermion theory. This problem is resolved by realizing that this theory can only appear at the boundary of the $\nu = 1$ integer quantum Hall effect of fermions in $2 + 1$ dimensions. In this setup the physical interpretation of charge non-conservation in the chiral Dirac fermion theory is that in the presence of the external field A_μ charge will flow from the bulk of the quantum Hall phase to the boundary. So charge is not conserved at the boundary, but the total charge in the bulk plus boundary is conserved. We chose this example because later in Chapters 4 and 5 of the thesis we study NLSMs with Wess-Zumino term which have a similar anomaly and which can appear at the boundary of the bosonic analogue of the integer quantum Hall state, which is the BIQH state introduced earlier in this section.

Our example of a theory with a global anomaly is a single Dirac fermion in $0 + 1$ dimensions. At the classical level this theory has $U(1)$ charge conservation symmetry and a discrete $\mathbb{Z}_2$ *charge-conjugation* symmetry. However, this theory has a global 't Hooft anomaly when it is coupled to a background gauge field A_μ for the $U(1)$ symmetry. In this case the anomaly implies that the charge of the states in this theory can be fractional, depending on which regularization procedure is used. For example, in one regularization scheme we will see that the states of the theory have charge $\pm\frac{1}{2}$ even though the Dirac fermion carries charge 1 under the $U(1)$ symmetry. This theory can occur at the edge of a one-dimensional chain of fermions in a SPT phase protected by $U(1)$ charge conservation symmetry and discrete $\mathbb{Z}_2$ charge-conjugation symmetry (the model studied in Ref. [5] can be thought of as a field theory description of such a chain of fermions). We chose this specific example because in Chapter 5 we study an exact bosonic analogue of this anomaly in NLSMs with Wess-Zumino term in $0 + 1$ dimensions. As we emphasize in that Chapter, the global anomaly of the Dirac fermion and the global anomaly in the bosonic model have the same mathematical origin and they both involve the Atiyah-Patodi-Singer *eta invariant* [71] for the Dirac operator in $0+1$ spacetime dimensions. We would like to emphasize here that this deep mathematical connection between the bosonic and fermionic anomalies was only exposed in our work because we solved the bosonic problem in full generality by applying the equivariant localization technique.

1.3.1 Perturbative anomalies

The first kind of anomaly that we consider is known as a *perturbative* anomaly. In this kind of anomaly the partition function for the field theory coupled to the background gauge field is not invariant under any gauge transformations, including infinitesimal gauge transformations. Here we illustrate this kind of anomaly via a simple example: a single chiral Dirac fermion in $1+1$ dimensions. This example is also physically relevant as this chiral fermion theory is the edge theory for the $\nu = 1$ integer quantum Hall effect in $2+1$ dimensions and, as we will see, the anomaly of the chiral Dirac fermion is a signature of the topological nature of the bulk quantum Hall phase. The anomaly of the chiral Dirac fermion theory is very similar to the original axial anomaly studied in Refs. [62, 63]. There is, however, a certain subtlety which arises in giving a proper definition of the fermion path integral for a *chiral* Dirac fermion as compared with the non-chiral case. We do not discuss it here but just mention that the correct definition of the path integral for the chiral Dirac fermion (which is needed for the path integral derivation of the anomaly following Fujikawa [64]) was given in Ref. [72].

The chiral Dirac fermion is, roughly speaking, one half of the full Dirac fermion in $1+1$ dimensions. Recall that a Dirac fermion Ψ in $1+1$ spacetime dimensions is a two-component spinor. The action for this Dirac fermion is constructed with the help of a set of gamma matrices γ^μ, $\mu = 0, 1$, which obey the Clifford algebra $\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}$, where $\eta = \text{diag}(1, -1)$ as a matrix (the "mostly minus" convention). Using these gamma matrices the action for a massless Dirac fermion takes the form

$$S[\Psi, \overline{\Psi}] = \int d^2x \, i\overline{\Psi}\gamma^\mu\partial_\mu\Psi \ , \tag{1.3.1}$$

where $\overline{\Psi} = \Psi^\dagger\gamma^0$ is usually called the Dirac adjoint of Ψ.

In the Weyl, or chiral, basis for the gamma matrices we have $\gamma^0 = \sigma^x$, $\gamma^1 = \sigma^y$ and $\overline{\gamma} = \gamma^0\gamma^1 = \sigma^z$, where $\sigma^{x,y,z}$ are the Pauli matrices and we introduced the chirality matrix $\overline{\gamma}$ (usually called γ^5 in $3+1$ dimensions). If we write Ψ in components as $\Psi = (\psi_R, \psi_L)^T$, then we find that the action written in the chiral basis has the form

$$S[\Psi, \overline{\Psi}] = \int d^2x \, \left[i\psi_R^\dagger(\partial_0 + \partial_1)\psi_R + i\psi_L^\dagger(\partial_0 - \partial_1)\psi_L \right] \ . \tag{1.3.2}$$

In particular, the two components of Ψ decouple, and the classical equations of motion imply that ψ_R moves to the right and ψ_L moves to the left (and this is why we used the subscripts R and L for the components of Ψ in this basis).

The chiral Dirac fermion theory is then a theory consisting of only one of the two chiral components of the full

Dirac fermion, say a left-moving fermion ψ_L, and this theory has an action

$$S[\psi_L, \psi_L^\dagger] = \int d^2x\ i\psi_L^\dagger(\partial_0 - \partial_1)\psi_L\ . \tag{1.3.3}$$

This action has a global $U(1)$ symmetry due to the fact that it is invariant under the transformation

$$\psi_L \to e^{i\alpha}\psi_L \tag{1.3.4}$$

for any *constant* phase α. We can then consider coupling the action to a background gauge field A_μ for this $U(1)$ symmetry. The minimally coupled action takes the form

$$S[\psi_L, \psi_L^\dagger, A] = \int d^2x\ i\psi_L^\dagger(\partial_0 - iA_0 - \partial_1 + iA_1)\psi_L\ , \tag{1.3.5}$$

and this action is invariant under the $U(1)$ gauge transformation

$$\psi_L \quad \to \quad e^{i\alpha(x)}\psi_L \tag{1.3.6}$$

$$A_\mu \quad \to \quad A_\mu + \partial_\mu\alpha(x)\ , \tag{1.3.7}$$

where the phase $\alpha(x)$ is now an arbitrary function of spacetime. Thus, at this point we have established the following two properties: (i) the classical action for the chiral Dirac fermion has a global $U(1)$ symmetry, and (ii) the classical action can be coupled to a background gauge field for the $U(1)$ symmetry in such a way that the action coupled to the background field is invariant under a $U(1)$ gauge transformation.

We now come to the anomaly. Suppose we compute the partition function $Z[A]$ of the chiral Dirac fermion coupled to A_μ. This partition function is formally given by the path integral

$$Z[A] = \int [d\psi_L][d\psi_L^\dagger]e^{iS[\psi_L,\psi_L^\dagger,A]}\ . \tag{1.3.8}$$

Naively, one expects this partition function to inherit the gauge invariance of the classical action $S[\psi_L, \psi_L^\dagger, A]$, i.e., one expects that

$$Z[A + d\alpha] = Z[A] \tag{1.3.9}$$

for any gauge transformation function α[4]. However, due to the need for regularization in quantum field theory, the

[4]Here we are using differential form notation in which $A = A_\mu dx^\mu$ and $d\alpha = \partial_\mu\alpha\ dx^\mu$.

partition function $Z[A]$ is not gauge invariant and instead transforms under a gauge transformation as

$$Z[A+d\alpha] = Z[A]e^{i\frac{1}{4\pi}\int d^2x\ \alpha\epsilon^{\mu\nu}\partial_\mu A_\nu}\ . \tag{1.3.10}$$

Therefore, the chiral Dirac fermion is an anomalous theory as the partition function $Z[A]$ does not inherit all the symmetries of the classical action $S[\psi_L, \psi_L^\dagger, A]$. It is also important to note that the anomaly is present even for infinitesimal gauge transformations $\alpha(x)$, and so this anomaly is indeed a perturbative anomaly according to our definition above.

We now comment on the relation between this anomaly and the $\nu = 1$ integer quantum Hall effect. Before we discuss the relation, let us first rewrite the gauge transformation of $Z[A]$ using differential form notation as

$$Z[A+d\alpha] = Z[A]e^{i\frac{1}{4\pi}\int \alpha F}\ , \tag{1.3.11}$$

where we wrote

$$\begin{aligned}
\int d^2x\ \alpha\epsilon^{\mu\nu}\partial_\mu A_\nu &= \int dx^0 \wedge dx^1\ \alpha\epsilon^{\mu\nu}\partial_\mu A_\nu \\
&= \int dx^\mu \wedge dx^\nu \alpha\partial_\mu A_\nu \\
&= \int \alpha dA \\
&= \int \alpha F\ ,
\end{aligned} \tag{1.3.12}$$

where $F = dA$ is the field strength for the gauge field A_μ with components $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. Now we are ready to discuss the relation to the integer quantum Hall effect.

The bulk electromagnetic response of the $\nu = 1$ integer quantum Hall phase is described by the Chern-Simons effective action

$$S_{CS}[A] = \frac{1}{4\pi}\int_{\mathcal{M}} A \wedge dA = \frac{1}{4\pi}\int_{\mathcal{M}} d^3x\ \epsilon^{\mu\nu\lambda}A_\mu\partial_\nu A_\lambda\ , \tag{1.3.13}$$

where $\mathcal{M}$ is the spacetime manifold. The current which flows in the system as a response to the field A_μ is obtained by taking a functional derivative of this action with respect to A_μ,

$$j^\mu = \frac{\delta S_{CS}[A]}{\delta A_\mu} = \frac{1}{2\pi}\epsilon^{\mu\nu\lambda}\partial_\nu A_\lambda\ . \tag{1.3.14}$$

In particular, this effective action captures the usual Hall response to an applied electric field. We should also make a comment here about the units we are using. Here (and in our discussion of the Dirac fermion) we use units in which

$\hbar = e = 1$. If we restore these fundamental constants then we find that ($h = 2\pi\hbar$)

$$j^{\mu} = \frac{e^2}{h}\epsilon^{\mu\nu\lambda}\partial_{\nu}A_{\lambda} \ , \tag{1.3.15}$$

and so we see that the Hall conductance is equal to $\frac{e^2}{h}$, exactly as we expect for the $\nu = 1$ integer quantum Hall state.

Now we are ready to explain the connection between the bulk Chern-Simons action $S_{CS}[A]$ and the anomaly of the chiral Dirac fermion. Suppose we do a gauge transformation $A \to A + d\alpha$ in the bulk Chern-Simons action. If the spacetime manifold $\mathcal{M}$ has no boundary then the Chern-Simons action is gauge invariant, but if $\mathcal{M}$ does have a boundary, then we find that

$$S_{CS}[A + d\alpha] = S_{CS}[A] + \frac{1}{4\pi}\int_{\partial\mathcal{M}} \alpha F \ , \tag{1.3.16}$$

where $\partial\mathcal{M}$ denotes the boundary of $\mathcal{M}$. We learn from this that if the $\nu = 1$ integer quantum Hall state is placed on a spacetime with a boundary, then the principle of gauge invariance demands that there be some degrees of freedom on the boundary which can cancel the gauge variation of the bulk Chern-Simons action [73, 74]. In addition, by comparing with Eq. (1.3.11) we can see that the anomaly of the (left-moving) chiral Dirac fermion is exactly right to cancel the gauge variation of the Chern-Simons term. Thus, we learn that the chiral Dirac fermion is a consistent theory for the edge of the $\nu = 1$ integer quantum Hall effect. It is consistent because with a left-moving chiral fermion on the boundary and a Chern-Simons term in the bulk, the bulk and boundary taken together are completely gauge invariant.

This phenomenon of anomaly cancellation between an anomalous theory in a particular spacetime dimension and a topological term in one higher dimension has been dubbed *anomaly inflow* and was first investigated by Callan and Harvey [75]. This concept has also been indispensable in the study of SPT phases, where the topological properties of the bulk of the phase can be understood from the anomaly of the boundary theory [66–70]. We now move on to an explanation of the concept of a global anomaly.

1.3.2 Global anomalies

We now introduce the second kind of anomaly, known as a *global* anomaly. The word global here refers to the fact that the anomaly can only be exposed by considering gauge transformations which are topologically nontrivial. These kinds of gauge transformations are called *large* gauge transformations. In the examples of global 't Hooft anomalies (which are the examples relevant for the study of SPT phases) the theory under consideration has both a discrete and a continuous symmetry, and the anomaly manifests itself as a conflict between these two symmetries in the quantized theory. To see the anomaly one should first couple the theory to a background gauge field for the continuous symmetry and then compute the partition function for the theory. One then finds that this partition function can be made to be invariant under either the discrete symmetry of the original theory, or large gauge transformations of the background gauge field, but not both. To make the theory invariant under one or the other of these symmetries one can add suitable local functionals of the background gauge field to the original Lagrangian. However, there is no local functional of the background gauge field that can make the final partition function invariant under both of these symmetries. As in the perturbative case, the anomaly can be traced back to the need to regularize the quantum field theory. However, unlike the pertubative case, here the anomaly is only present for large gauge transformations, which are gauge transformations that cannot be continuously deformed to a trivial gauge transformation (i.e., to no transformation at all). The first example of a global anomaly was due to Witten [65], but that particular example is not a 't Hooft anomaly because the gauge field in Witten's example is dynamical (i.e., one integrates out the gauge field to compute the partition function of the quantum field theory) instead of a background gauge field.

Our example of a theory with a global anomaly is the theory of a single Dirac fermion in $0+1$ spacetime dimensions. Global anomalies in this theory were first considered in Ref. [76]. In addition, this theory can be considered as a toy model for a more interesting global anomaly, which is the *parity* anomaly of the massless Dirac fermion in $2+1$ dimensions [58–60]. The reader should note that our presentation differs slightly from that of Ref. [76] because we use a different regularization procedure which allows us to connect our review of this anomaly in this section with our calculation of a global anomaly in a bosonic theory in Chapter 5.

The action for the Dirac fermion in $0+1$ dimensions has the form

$$S[\psi,\psi^\dagger] = \int_0^T dt\; i\psi^\dagger\partial_t\psi\ , \tag{1.3.17}$$

where ψ is a one-component fermion. To study the theory in as concrete a manner as possible we impose anti-periodic boundary conditions on the fermion in the time direction, $\psi(t+T) = -\psi(t)$, where T is the time interval that we consider. This theory has a global $U(1)$ symmetry and a discrete $\mathbb{Z}_2$ charge-conjugation symmetry. The fermion ψ transforms under these two symmetries as

$$U(1):\ \psi \to e^{i\alpha}\psi \tag{1.3.18}$$

for a constant phase α, and

$$\mathbb{Z}_2:\ \psi \to \psi^\dagger\ . \tag{1.3.19}$$

To expose the anomaly in the theory we couple it to a background gauge field A_t for the $U(1)$ symmetry and then compute the partition function of the theory coupled to A_t. The action for the gauged theory is

$$S[\psi,\psi^\dagger,A] = \int_0^T dt\; i\psi^\dagger(\partial_t - iA_t)\psi\ . \tag{1.3.20}$$

One can check that this action is invariant under the $U(1)$ gauge transformation

$$\begin{aligned} \psi &\rightarrow e^{i\alpha(t)}\psi && (1.3.21) \\ A_t &\rightarrow A_t + \partial_t \alpha(t) && (1.3.22) \end{aligned}$$

for any gauge function $\alpha(t)$, and it is also invariant under the discrete $\mathbb{Z}_2$ charge-conjugation transformation[5]

$$\begin{aligned} \psi &\rightarrow \psi^\dagger && (1.3.23) \\ A_t &\rightarrow -A_t \,. && (1.3.24) \end{aligned}$$

In discussing these symmetries it is important to distinguish between two kinds of gauge transformation functions $\alpha(t)$. We call a gauge transformation a *small* gauge transformation if $\int_0^T dt\, \partial_t \alpha(t) = \alpha(T) - \alpha(0) = 0$. In other words, a small gauge transformation does not wind around the time direction. Next, we define a *large* gauge transformation as one which does wind in the time direction, $\int_0^T dt\, \partial_t \alpha(t) \neq 0$. In addition, we require that the exponential $e^{i\int_0^T dt\, A_t}$ is gauge invariant, which implies that all gauge transformation functions $\alpha(t)$ satisfy $\alpha(T) - \alpha(0) = 2\pi k$ for some integer k. This requirement is equivalent to the statement that the gauge group is *compact* $U(1)$.

We can always decompose a particular gauge field A_t as

$$A_t = \overline{A}_t + \partial_t \beta(t) \tag{1.3.25}$$

where

$$\overline{A}_t = \frac{1}{T}\int_0^T dt\, A_t \tag{1.3.26}$$

is the time average of the gauge field A_t and $\beta(t)$ satisfies $\int_0^T dt\, \partial_t \beta(t) = 0$. In the langauge of differential forms this is equivalent to the statement that on the circle S^1 (which corresponds to the time direction) any one-form $A = A_t dt$ can be decomposed into exact and non-exact pieces. In this case $\overline{A}_t dt$ is the non-exact piece and $\partial_t \beta dt = d\beta$ is the exact piece. This decomposition is important because we can then do a small gauge transformation in the action to remove the exact piece of A_t, so that the action for the fermion coupled to A_t can be reduced to the simpler form

$$S[\psi, \psi^\dagger, A] = \int_0^T dt\, i\psi^\dagger(\partial_t - i\overline{A}_t)\psi \,. \tag{1.3.27}$$

Finally, in this decomposition the large gauge transformations consistent with the compactness condition on the gauge

[5]To see that the action has this discrete symmetry one should recall that ψ and $\psi^\dagger$ should be treated as Grassmann-valued (i.e., anti-commuting) fields in the classical action, and so $\psi^\dagger\psi = -\psi\psi^\dagger$.

field take the form

$$\overline{A}_t \to \overline{A}_t + \frac{2\pi k}{T}\ ,\ k \in \mathbb{Z}\ . \tag{1.3.28}$$

We are now ready to compute the partition function $Z[A]$ for this theory and identify its anomalous properties. Formally, we have to compute the path integral

$$Z[A] = \int [d\psi][d\psi^\dagger] e^{iS[\psi,\psi^\dagger,A]}\ , \tag{1.3.29}$$

which evaluates to the determinant of the Dirac operator $i(\partial_t - i\overline{A}_t)$,

$$Z[A] = \text{det}[i(\partial_t - i\overline{A}_t)]\ . \tag{1.3.30}$$

This determinant, however, requires regularization to be well-defined, and this regularization procedure will ultimately be responsible for the anomaly. One way to regularize this determinant is to define the amplitude and phase of the determinant using *zeta* and *eta* functions, respectively. We discuss this regularization procedure in detail in Chapter 5 and in Appendix D and so here we limit ourselves to one remark about this method[6]. The zeta and eta functions depend only on the eigenvalues of the Dirac operator, and so this regularization method preserves (large and small) gauge invariance since the eigenvalues of the Dirac operator are gauge invariant. This method yields the partition function

$$Z[A] = 1 + e^{i\overline{A}_t T} = 1 + e^{i\int_0^T dt\ A_t}\ . \tag{1.3.31}$$

This answer for the partition function is invariant under small $U(1)$ gauge transformations as well as the large $U(1)$ gauge transformations in which $\overline{A}_t \to \overline{A}_t + \frac{2\pi k}{T}$, $k \in \mathbb{Z}$. Note, however, that this form of $Z[A]$ is *not* invariant under the charge-conjugation transformation $A_t \to -A_t$, and this can be traced back to the fact that the eta invariant for $i(\partial_t - i\overline{A}_t)$ is not invariant under this transformation.

We see that the calculation of the partition function using the zeta and eta function method has produced a partition function which retains the large $U(1)$ gauge symmetry of the original action $S[\psi,\psi^\dagger,A]$, but not the charge-conjugation symmetry. We can try to restore the charge-conjugation symmetry by adding some local functional $F[A]$ of A_t to the action. This modifies the partition function to

$$Z'[A] = e^{iF[A]} Z[A]\ . \tag{1.3.32}$$

[6] In comparing the result of this section to the result of Chapter 5 the reader should keep in mind that in this section we consider the eigenvalues of the Dirac operator with anti-periodic boundary conditions, while in Chapter 5 we consider the same operator but with periodic boundary conditions.

To satisfy $Z'[-A] = Z'[A]$ the functional $F[A]$ must satisfy the equation

$$F[-A] - F[A] = \overline{A}_t T \ . \tag{1.3.33}$$

The simplest possible solution of this is $F[A] = -\frac{1}{2}\overline{A}_t T$, and this solution yields the partition function

$$Z'[A] = 2\cos\left(\frac{\overline{A}_t T}{2}\right) \ . \tag{1.3.34}$$

However, this version of the partition function is no longer invariant under large $U(1)$ gauge transformations, for example $Z'[A]$ will change sign under $\overline{A}_t \to \overline{A}_t + \frac{2\pi}{T}$. The conclusion is that it is impossible to regularize the Dirac fermion theory in such a way that the partition function retains both the charge-conjugation symmetry and the invariance under large $U(1)$ gauge transformations which is present in the classical action $S[\psi, \psi^\dagger, A]$. We also mention here that the local functional $F[A] = -\frac{\overline{A}_t T}{2} = -\frac{1}{2}\int_0^T dt\ A_t$ is actually the $0+1$ dimensional version of a Chern-Simons term for A_t, but with the level of this Chern-Simons term equal to $-\frac{1}{2}$.

We mentioned above that a physical consequence of this anomaly is the presence of fractional charge in the theory. We now explain this in more detail. In fact, we will see that the fractional charge is only present when we choose the regularization procedure which preserves the charge-conjugation symmetry but lacks the large $U(1)$ gauge invariance. To understand this point it is actually easier to think about the theory from the viewpoint of canonical quantization. In the canonical approach ψ and $\psi^\dagger$ obey the anti-commutation relation $\{\psi, \psi^\dagger\} = 1$ and the theory consists of only two states. The first state is the vacuum state $|0\rangle$ annihilated by ψ and the second state is $|1\rangle = \psi^\dagger|0\rangle$ in which a single fermion is present. The charge operator Q in this theory can be read off from the gauged classical action as it is simply equal to the terms in the action which are multiplied by A_t. For the regularization which preserves large $U(1)$ gauge invariance the gauged action has the form shown in Eq. (1.3.20) and so the charge operator is just $Q = \psi^\dagger\psi$ and the two states $|0\rangle$ and $|1\rangle$ have charge 0 and 1, respectively. Thus, in the gauge invariant regularization of the theory all states have integer charge. Next, for the regularization of the theory which preserves the charge-conjugation symmetry the full action included the functional $F[A] = -\frac{\overline{A}_t T}{2} = -\frac{1}{2}\int_0^T dt\ A_t$ and so the charge operator in this regularization is $Q = \psi^\dagger\psi - \frac{1}{2}$. Then in this regularization the states $|0\rangle$ and $|1\rangle$ have charge $-\frac{1}{2}$ and $\frac{1}{2}$, respectively, and so the states have fractional charge. It is worth emphasizing here that the presence of fractional charge is quite surprising as the Dirac fermion field ψ carries a charge of 1 under the $U(1)$ symmetry.

This concludes our discussion on anomalies, and we now move on to the main Chapters of the thesis.

Chapter 2

Swimming at low Reynolds number in fluids with odd, or Hall, Viscosity[1]

2.1 Introduction

The theory of swimming in classical fluids at low Reynolds number [24, 77] is remarkable because of the connections it makes between seemingly disparate fields [25]. For example, the motion of swimmers with cyclic swimming strokes is determined purely from classical fluid dynamics, but it can be re-cast into an elegant geometric formulation reminiscent of Berry's phase physics and gauge fields [25, 78, 79]. In fact, the motion of tiny organisms in fluids with high viscosity can be captured by a "gauge-theory" of shapes. Since the initial work on the geometric formulation of swimming there have been generalizations to swimmers in quantum fluids [80] and even to swimmers in fluids on curved spaces [81, 82]. The theory has also been successfully applied in practice to describe the swimming of robots [83] and microbots [84, 85].

In this Chapter we focus on swimmers in 2D fluids with broken time-reversal symmetry, for example, fluids in magnetic fields or rotating fluids. We are not interested in the specific source of time-reversal breaking, but instead just consider a classical fluid with a microscopic source of local angular momentum (on a much smaller scale than the size of the swimmer) that gives rise to a non-vanishing "odd" viscosity coefficient [22, 23] in addition to the usual isotropic viscosity coefficients. The odd viscosity is an off-diagonal viscosity term that is dissipationless and produces forces perpendicular to the direction of the fluid flow. It can have a quantum mechanical origin in, for example, systems exhibiting the quantum Hall effect [22, 23, 86–94], or a classical origin in plasmas at finite-temperature [95]. In the quantum Hall setting the odd viscosity is usually known as the Hall viscosity. It is also sometimes referred to as Lorentz shear stress.

We will not focus on the microscopic origin of the odd viscosity coefficient, but only assume it to be non-vanishing in conjunction with the usual viscosity coefficients. From this assumption we will determine the motion of swimmers at low Reynolds number in the presence of odd viscosity. Specifically, we will consider the problem of swimmers with circular boundaries that move via deformations of their boundaries analogous to the nearly-circular swimmers in Ref. [25, 78]. We find a general result that connects the torque on a swimmer to the rate of area change of the swimmer

[1]This Chapter is reproduced from Matthew F. Lapa and Taylor L. Hughes, Phys. Rev. E 89, 043019 (2014). This paper is also cited as Ref. [26] in the References section of the thesis.

with a proportionality constant given by the odd viscosity. We use our results to give examples of swimmer motion due to cyclic circular deformations and compare cases where the conventional and odd viscosities each dominate. This Chapter is organized as follows: we first review the geometric formulation of swimming and the appearance of odd viscosity in 2D fluids with broken time-reversal symmetry. We then go on to derive the general consequences of the odd viscosity on swimmers and then give explicit examples of model swimming strokes that illustrate some differences between fluids with vanishing and non-vanishing odd viscosity. In the last section we consider reciprocal swimming strokes and show how the famous Scallop theorem of low Reynolds number swimming carries over to the case of fluids with odd viscosity. Finally, we have some appendices which collect derivations of the more technical results.

2.2 Review of Geometric Formulation of the Swimming Problem

We begin by reviewing the geometric formulation of the problem of swimming at low Reynolds number developed by Shapere and Wilczek [25, 78]. The instantaneous rigid motion (translation and rotation) of a swimmer is determined by the condition that the swimmer not be able to exert a net force or torque on itself, and the condition that the fluid velocity vanishes at infinity.

We should first explain why the problem of swimming at low Reynolds number can be formulated in a purely geometric way, independent of the mass of the swimmer or the speed of the swimming stroke (assuming the speed of the stroke is still small enough so that there is no appreciable momentum transfer to the fluid). Recall that the Reynolds number, which is associated with a viscous fluid and an object in motion in that fluid, is a ratio of the inertial and viscous forces on that object (we are not yet considering systems with odd viscosity so in this sentence the word "viscous" refers to the traditional dissipative (even) viscosity of the fluid). If η^e is the even viscosity coefficient, V is a typical speed of the fluid flow, L is a characteristic dimension of the swimming object, and ρ is the density of the fluid, then the Reynolds number can be expressed as

$$\mathrm{Re} = \frac{\rho V L}{\eta^e} \ . \tag{2.2.1}$$

The low Reynolds number regime can be interpreted as the regime where the momentum density of the fluid, ρV, is negligible compared to the scale η^e/L.

At low Reynolds number the drag force on the swimmer is proportional to its velocity. This means that if the swimmer stops its stroke and just coasts through the fluid, its speed will decay exponentially until it comes to a stop. In the low Reynolds number regime this exponential decay is so fast that the motion of the swimmer at any given time can be considered to be completely independent of what the swimmer was doing at all previous times [24]. The motion of the swimmer at time t depends only on its shape and the velocity of its surface at time t. With these remarks in mind we can move on to discuss the geometric theory of swimming at low Reynolds number.

In two dimensions, for swimmers modeled as the interior of deformed circles, we can represent the swimming stroke (the motion of the boundary of the swimmer) by a time-dependent complex function $S_0(\sigma, t)$, $\sigma = e^{i\theta}$, whose real and imaginary parts give the x and y positions of the point on the swimmer described by the parameter $\theta \in [0, 2\pi)$ at the time t. When we want to emphasize the dependence of $S_0(\sigma, t)$ on the real parameter θ instead of the complex parameter σ (as we do in Appendix A.2) we call it $S_0(\theta, t)$ instead.

The function $S_0(\sigma, t)$ lives in a space of "un-located" shapes, which can be obtained from the space of "located" shapes by partitioning it into equivalence classes $[S_0(\sigma, t)]$ containing all shapes differing only by a rigid motion. The location and orientation of the swimmer in real space is specified by a rigid motion $\mathcal{R}(t)$ acting on a representative of the equivalence class $[S_0(\sigma, t)]$, the simplest choice being $S_0(\sigma, t)$ itself:

$$S(\sigma, t) = \mathcal{R}(t) S_0(\sigma, t) \; . \tag{2.2.2}$$

To take an example, $S_0(\sigma, t)$ might be the representative of $[S_0(\sigma, t)]$ with its centroid at the origin and a distinguishing feature of the shape aligned with the x-axis at time $t = 0$.

To be concrete, let us encode the translation and rotation represented by $\mathcal{R}(t)$ into a 3×3 matrix and let this matrix act on $S_0(\sigma, t)$ represented as a three-dimensional vector with third entry equal to one,

$$\mathcal{R}(t) S_0(\theta, t) = \begin{pmatrix} \cos(\Theta) & \sin(\Theta) & X \\ -\sin(\Theta) & \cos(\Theta) & Y \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \mathrm{Re}[S_0(\sigma, t)] \\ \mathrm{Im}[S_0(\sigma, t)] \\ 1 \end{pmatrix} \; , \tag{2.2.3}$$

where (X, Y) and Θ are the vector and angle representing the translation and rotation effected by $\mathcal{R}(t)$. The matrix $\mathcal{R}(t)$ is determined by integrating the equation

$$\frac{d\mathcal{R}(t)}{dt} = \mathcal{R}(t) \mathcal{A}(t) \; , \tag{2.2.4}$$

where the matrix $\mathcal{A}(t)$ determines the infinitesimal rigid motion of the swimmer during a time dt in the sense that $\mathcal{A}(t)\ dt$ *is* the rigid motion of the swimmer during the interval dt. The matrix $\mathcal{A}(t)$ is completely determined by the requirements that the net force and torque on the swimmer vanish and that the fluid velocity goes to zero at infinity. To determine the swimming path we need to find $\mathcal{A}(t)$ for a given swimming stroke and then integrate Eq. 2.2.4.

Integrating this equation gives the solution for the rigid motion $\mathcal{R}(t)$,

$$\mathcal{R}(t) = \mathcal{R}(0)\bar{P}\text{exp}\left[\int_0^t \mathcal{A}(t')dt'\right] , \tag{2.2.5}$$

where $\bar{P}$ denotes a reverse path-ordering operation. Explicitly, we have

$$\bar{P}\text{exp}\left[\int_0^t \mathcal{A}(t')dt'\right] = I + \int_0^t \mathcal{A}(t_1)dt_1 + \int_0^t \left(\int_0^{t_1} \mathcal{A}(t_2)\mathcal{A}(t_1)dt_2\right) dt_1 + \dots \tag{2.2.6}$$

so the matrix $\mathcal{A}(t_i)$ with the latest time t_i appears furthest to the right in each integral, which is the reverse of the usual path ordering operation where the latest time goes furthest to the left in each integral. We show how this integration is carried out numerically in Appendix A.1.

To see how the idea of a gauge theory of shapes enters we first note that the choice of a representative from the equivalence class $[S_0(\sigma,t)]$ is analogous to a choice of gauge, and the matrix $\mathcal{A}(t)$ plays the role of a gauge potential. If we choose a different representative $\tilde{S}_0(\sigma,t)$, related to $S_0(\sigma,t)$ by a rigid motion $U(t)$ (we can choose a different representative at each time t),

$$\tilde{S}_0(\sigma,t) = U(t)S_0(\sigma,t) , \tag{2.2.7}$$

then the requirement that the rigid motion of the swimmer in real space remain unchanged leads to the transformation law for $\mathcal{R}(t)$

$$\mathcal{R}(t) \to \mathcal{R}'(t) = \mathcal{R}(t)U^{-1}(t) . \tag{2.2.8}$$

The fact that the transformed gauge potential must satisfy the new differential equation

$$\frac{d\mathcal{R}'(t)}{dt} = \mathcal{R}'(t)\mathcal{A}'(t) \tag{2.2.9}$$

yields the familiar transformation law

$$\mathcal{A}(t) \to \mathcal{A}'(t) = U(t)\mathcal{A}(t)U^{-1}(t) + U(t)\frac{dU^{-1}(t)}{dt} , \tag{2.2.10}$$

which shows that $\mathcal{A}(t)$ does indeed transform like a gauge potential.

We can also represent $\mathcal{A}(t)$ in the form of a 3×3 matrix,

$$\mathcal{A}(t) = \begin{pmatrix} 0 & \omega & V_x \\ -\omega & 0 & V_y \\ 0 & 0 & 0 \end{pmatrix} \tag{2.2.11}$$

where (V_x, V_y) and ω are the instantaneous linear and angular velocity of the swimmer (so $\mathcal{A}(t)$ is in the Lie Algebra of rigid motions). Sometimes we will refer to the translational and rotational parts $\mathcal{A}_{tr}$ and $\mathcal{A}_{rot}$ of the gauge potential, defined by

$$\mathcal{A}_{tr} = V_x + iV_y \tag{2.2.12a}$$

$$\mathcal{A}_{rot} = \omega \, . \tag{2.2.12b}$$

The components of $\mathcal{A}(t)$ can be completely determined by solving the equations of motion for Stokes flow of the viscous fluid surrounding the swimmer, subject to no-slip boundary conditions at the surface of the swimmer. Now that we have reviewed the geometric formulation of swimming we will introduce the concept of the odd viscosity in time-reversal breaking fluids.

2.3 Odd Viscosity

We now review the basic definition of odd viscosity and the derivation of the isotropic odd viscosity contribution to the fluid stress tensor in two dimensions. Throughout this section we follow the presentation of Ref. [23] where most of these details were first worked out.

The general linear relation between the fluid stress tensor T_{ij} and the rate of strain tensor $v_{ij} = \frac{1}{2}(\partial_j v_i + \partial_i v_j)$ (v_i are the components of the fluid velocity vector $\mathbf{v}$) is of the form

$$T_{ij} = \eta_{ijkl} v_{kl} \, . \tag{2.3.1}$$

The symmetry of the stress and rate of strain tensors imply the symmetry of the viscosity tensor η_{ijkl} under the exchanges $i \leftrightarrow j$ and $k \leftrightarrow l$, but in general η_{ijkl} can contain terms which are symmetric or anti-symmetric under the exchange of the pair of indices $\{ij\}$ with the pair of indices $\{kl\}$. We can always split η_{ijkl} into parts which are even and odd under such an exchange by writing $\eta_{ijkl} = \eta^e_{ijkl} + \eta^o_{ijkl}$.

To extract the isotropic contribution to η^o_{ijkl} it is convenient to use a simple basis for representing a real, 4th-rank tensor that is symmetric under exchange of its first two and second two indices. One such basis is provided by the tensor products

$$\sigma^a \otimes \sigma^b, \ a, b \in \{0, 1, 3\} \tag{2.3.2}$$

of the Pauli matrices σ^1, σ^3 and the 2×2 identity matrix σ^0, where we have been careful to only use the symmetric matrices. We can expand the viscosity tensor as

$$\eta_{ijkl} = \sum_{a,b=0,1,3} \eta_{ab}\sigma^a_{ij}\sigma^b_{kl} \tag{2.3.3}$$

and then identify the odd part as

$$\eta^o_{ijkl} = \sum_{a\neq b} \eta^o_{ab}(\sigma^a_{ij}\sigma^b_{kl} - \sigma^b_{ij}\sigma^a_{kl}) \ . \tag{2.3.4}$$

In two dimensions the generator of spatial rotations is $i\sigma^2$, where σ^2 is the second Pauli matrix. In an isotropic fluid the viscosity tensor must commute with $\sigma^2 \otimes \sigma^2$ to be rotationally invariant. Using the familiar commutation and anti-commutation relations for the Pauli matrices, and the fact that all matrices commute with the identity σ^0, we find that in an isotropic fluid the odd part of the viscosity tensor must have the form

$$\eta^o_{ijkl} = \eta^o(\sigma^1_{ij}\sigma^3_{kl} - \sigma^3_{ij}\sigma^1_{kl}) \ , \tag{2.3.5}$$

where the single constant η^o is the coefficient of odd viscosity. Finally we can use the explicit expressions

$$\sigma^1_{ij} = \delta_{i1}\delta_{j2} + \delta_{i2}\delta_{j1} \tag{2.3.6a}$$

$$\sigma^3_{ij} = \delta_{i1}\delta_{j1} - \delta_{i2}\delta_{j2} \tag{2.3.6b}$$

for the elements of the Pauli matrices σ^1 and σ^3 to write down the form of the odd viscosity contribution to the stress tensor

$$\begin{aligned} T^o_{ij} &= \eta^o_{ijkl}v_{kl} \\ &= -2\eta^o(\delta_{i1}\delta_{j1} - \delta_{i2}\delta_{j2})v_{12} + \eta^o(\delta_{i1}\delta_{j2} + \delta_{i2}\delta_{j1})(v_{11} - v_{22}) \end{aligned} \tag{2.3.7}$$

which was first obtained in Ref. [23]. For comparison we also display the much more familiar even viscosity part of the stress tensor (for an incompressible fluid)

$$T^e_{ij} = 2\eta^e v_{ij} \ , \tag{2.3.8}$$

where η^e is the coefficient of even viscosity.

We see that diagonal elements of T^o_{ij} are proportional to off-diagonal elements of v_{ij} and off-diagonal elements of T^o_{ij} are proportional to diagonal elements of v_{ij}. This atypical relation between the elements of T^o_{ij} and v_{ij} has a number of non-intuitive consequences. For example, a circular object rotating in a fluid with odd viscosity will feel a pressure, directed either radially inwards or outwards depending on the sense of the rotation (see [23] and Fig. 2.1).

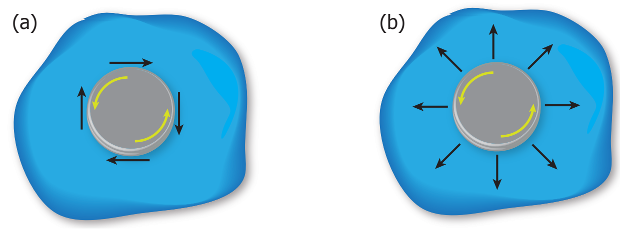

Figure 2.1: (a) In a fluid with even viscosity only, a rotating circle will feel a torque that opposes its rotation and is proportional to the coefficient of even viscosity η^e. (b) In a fluid with odd viscosity a rotating circle will also feel a pressure directed radially inwards or outwards (depending on the direction of the rotation) and proportional to the coefficient of odd viscosity η^o. The dependence of this pressure force on the direction of the rotation indicates that time-reversal symmetry is broken in systems with non-vanishing odd viscosity.

This is quite different from what would happen in a fluid with even viscosity only, where a rotating circle would feel a torque that opposes the rotation. The fact that the direction of the pressure force (radially inwards or outwards) on a circle rotating in an odd viscosity fluid depends on the sense of the rotation means that time-reversal symmetry is broken in systems with odd viscosity.

2.4 Equations of Motion, Force and Torque

In classical fluids with both even and odd viscosity Avron has shown (see [23]) that the equations of motion for incompressible Stokes flow (viscous force-dominated flow) are

$$\nabla(p - \eta^o \xi) = \eta^e \nabla^2 \mathbf{v} \tag{2.4.1a}$$

$$\nabla \cdot \mathbf{v} = 0\,, \tag{2.4.1b}$$

where p is the pressure, $\xi = (\nabla \times \mathbf{v}) \cdot \hat{\mathbf{z}}$ is the vorticity, and η^e and η^o are the coefficients of even and odd viscosity, respectively. We will refer to these equations as the "slow flow" equations, as that is what they are called in the usual case where only even viscosity is present. If $\eta^e \neq 0$, taking the curl of the first equation shows that the vorticity is a harmonic function, i.e., $\nabla^2 \xi = 0$. This means that the stream function ψ (which can be used here because the flow is incompressible), defined by $\mathbf{v} = \nabla \times (\psi \hat{\mathbf{z}})$, is a biharmonic function,

$$\nabla^2(\nabla^2 \psi) = 0\,. \tag{2.4.2}$$

In two dimensions we can package the velocity vector $\mathbf{v} = (v_1, v_2)$ into a complex variable $v = v_1 + iv_2$. The solution for v can then be expressed in the complex form (see [78])

$$v = \phi_1(z) - z\overline{\partial_z \phi_1(z)} + \overline{\phi_2(z)} \tag{2.4.3}$$

where $z = x + iy = Re^{i\varphi}$, the bar denotes complex conjugation and $\partial_z = \frac{1}{2}(\partial_1 - i\partial_2)$. The functions $\phi_1(z)$ and $\phi_2(z)$ are analytic functions (away from the point $z = 0$, which lies inside the swimmer) with the Laurent series expansions

$$\phi_1(z) = \sum_{k<0} a_k z^{k+1} \tag{2.4.4a}$$

$$\phi_2(z) = \sum_{k<-1} b_k z^{k+1} . \tag{2.4.4b}$$

To solve for the coefficients a_k and b_k we impose no-slip boundary conditions at the surface of the swimmer. Solving for these coefficients can be very difficult for general swimming strokes, so we will focus our attention on a class of simple swimmers introduced in Ref. [78] whose shapes are conformal maps of the circle of degree $\mathcal{D} = 2$. In Appendix A.3 we extend our results to swimmers that are conformal maps of the circle of degree $\mathcal{D} = 3$.

To calculate the force and torque on the swimmer we will need the stress tensor. We have seen in Section 2.3 that in the presence of odd viscosity the stress tensor gets an extra contribution. The full stress tensor is now

$$T_{ij} = -p\delta_{ij} + 2\eta^e v_{ij} - 2\eta^o(\delta_{i1}\delta_{j1} - \delta_{i2}\delta_{j2})v_{12} + \eta^o(\delta_{i1}\delta_{j2} + \delta_{i2}\delta_{j1})(v_{11} - v_{22}) . \tag{2.4.5}$$

The components of the odd-viscosity part of the stress tensor are

$$T^o_{11} = -\eta^o(\partial_2 v_1 + \partial_1 v_2) \tag{2.4.6a}$$

$$T^o_{12} = \eta^o(\partial_1 v_1 - \partial_2 v_2) \tag{2.4.6b}$$

$$T^o_{21} = \eta^o(\partial_1 v_1 - \partial_2 v_2) \tag{2.4.6c}$$

$$T^o_{22} = \eta^o(\partial_2 v_1 + \partial_1 v_2) . \tag{2.4.6d}$$

Since the fluid is incompressible, an application of the divergence theorem shows that the force and torque on the surface of the swimmer are the same as the force and torque on the fluid at infinity. Using this equivalence, the components of the force on the swimmer are

$$F_i = \lim_{R\to\infty} \int_0^{2\pi} (T_{ij} r_j) R d\varphi \tag{2.4.7}$$

and the torque on the swimmer is

$$N = \lim_{R\to\infty} \int_0^{2\pi} (\epsilon_{ij} r_i T_{jk} r_k) R^2 d\varphi \ . \tag{2.4.8}$$

In these formulas r_i are the components of the radial unit vector $\hat{\mathbf{r}} = \cos\varphi\hat{\mathbf{x}} + \sin\varphi\hat{\mathbf{y}}$ and the integral is taken over the circle at infinity.

Using these equations, and the components of the odd-viscosity part of the stress tensor, we can derive expressions for the odd-viscosity contribution to the force and torque on the swimmer. In complex form they are

$$F^o = \lim_{R\to\infty} -2\eta^o \oint_{\mathcal{C}} (\partial_{\bar{z}} v)\ d\bar{z} \tag{2.4.9}$$

and

$$N^o = \lim_{R\to\infty} -2\eta^o \ \mathrm{Re}\left\{ i \oint_{\mathcal{C}} z(\partial_z \bar{v})\ dz \right\} \tag{2.4.10}$$

where $\mathcal{C}$ is a circular contour of radius R (to be taken to infinity), and we have switched to a complex notation for the force, $F = F_1 + iF_2$ (the torque, being a scalar in 2D, is real).

Plugging in the velocity expansion (2.4.3) into these formulas gives

$$F^o = 0 \tag{2.4.11a}$$

$$N^o = -4\pi\eta^o \mathrm{Re}[b_{-2}] \ . \tag{2.4.11b}$$

In the next subsection we will show that the physical interpretation of this result is that the odd-viscosity contribution to the torque is proportional to the flux of the fluid at infinity (see Section 2.4.1). Previously it has been shown [78] that the even-viscosity contribution to the force and torque on the swimmer is given by

$$F^e = 0 \tag{2.4.12a}$$

$$N^e = 4\pi\eta^e \mathrm{Im}[b_{-2}] \ . \tag{2.4.12b}$$

The swimmer feels no net force (a generic result for Stokes flows in two dimensions [96]) and the total torque is

$$N = 4\pi(\eta^e \mathrm{Im}[b_{-2}] - \eta^o \mathrm{Re}[b_{-2}]) \ . \tag{2.4.13}$$

We can cancel the torque on the swimmer by having the swimmer rotate at a certain angular velocity ω. This uniquely determines the rotational part of the gauge potential. In dimensions $D > 2$ the translational part of the gauge potential can be determined by the condition that the net force on the swimmer vanish. In two dimensions, however, the net force vanishes identically [96] and so one must instead determine the translational part of the gauge potential by requiring that the fluid velocity vanish at infinity [97]. We discuss this condition in more detail in Section 2.6.

2.4.1 Physical Interpretation of the Torque Formula

The physical content of the formula (2.4.13) for the net torque on the swimmer can be better understood by looking at the relation of the coefficient b_{-2} to the circulation and flux of the fluid at infinity, denoted by $\Gamma(\infty)$ and $\Phi(\infty)$, respectively. We can express the circulation and flux of the fluid at infinity in the form of line integrals of the velocity around a large circle of radius R, to be taken to infinity. We have,

$$\Gamma(\infty) = \lim_{R\to\infty} \int_0^{2\pi} \mathbf{v} \cdot \hat{\boldsymbol{\varphi}} R d\varphi \tag{2.4.14}$$

and

$$\Phi(\infty) = \lim_{R\to\infty} \int_0^{2\pi} \mathbf{v} \cdot \hat{\mathbf{r}} R d\varphi \, . \tag{2.4.15}$$

Using the velocity expansion (2.4.3), we find

$$\Gamma(\infty) \quad = \quad -2\pi \mathrm{Im}[b_{-2}] \tag{2.4.16a}$$

$$\Phi(\infty) \quad = \quad 2\pi \mathrm{Re}[b_{-2}] \, . \tag{2.4.16b}$$

Using these expressions, the net torque on the swimmer can be rewritten in the form

$$N = -2\eta^e \Gamma(\infty) - 2\eta^o \Phi(\infty) \, . \tag{2.4.17}$$

The condition of vanishing torque in the different cases can then be interpreted in terms of zero circulation at infinity for even viscosity only, zero flux at infinity for odd viscosity only, or a proportionality between the flux and circulation at infinity when both types of viscosity are present.

2.5 Model Swimming Strokes and Area Formula

Following Ref. [78], we will begin by considering nearly circular swimmers with swimming strokes of the form

$$S_0(\sigma, t) = \alpha_0(t)\sigma + \alpha_{-2}(t)\sigma^{-1} + \alpha_{-3}(t)\sigma^{-2} \, , \tag{2.5.1}$$

where the $\alpha_i(t)$'s are coefficients which determine the time evolution of the swimming stroke. This kind of stroke is just a conformal map of degree $\mathcal{D} = 2$ from the unit circle to the complex z-plane. The absence of a term $\alpha_{-1}(t)$ "fixes the gauge" with respect to translations [78]. An important formula is the area of the swimmer at time t, which is given by

$$\begin{aligned} A(t) &= \frac{1}{2}\mathrm{Im}\left\{\oint \overline{S_0(\theta,t)}\, dS_0(\theta,t)\right\} \\ &= \frac{1}{2}\mathrm{Im}\left\{\int_0^{2\pi} \overline{S_0(\theta,t)}\, \frac{dS_0(\theta,t)}{d\theta} d\theta\right\} \ , \end{aligned} \tag{2.5.2}$$

which gives

$$A(t) = \pi(|\alpha_0|^2 - |\alpha_{-2}|^2 - 2|\alpha_{-3}|^2) \tag{2.5.3}$$

for the simple stroke (2.5.1). General swimmers represented by conformal maps of degree $\mathcal{D}$ have the form

$$S_0(\sigma,t) = \alpha_0(t)\sigma + \sum_{n=1}^{\mathcal{D}} \alpha_{-n}(t)\sigma^{-n} \tag{2.5.4}$$

and in Appendix A.3 we extend the swimming motion formulae to swimmers with $\mathcal{D} = 3$.

2.6 Solution for Translational and Rotational Motion of Swimmer

To determine the coefficients a_k and b_k in the velocity expansion (2.4.3) we need to conformally map the flow field back to the $\zeta = re^{i\theta}$ plane [78]. Recall that the shape of the swimmer $S_0(\sigma,t)$ is a conformal map in the other direction, from the unit circle $\sigma = e^{i\theta}$ in the ζ-plane to the z-plane. For general swimmers of the form (2.5.4) the conformal mappings between the ζ and z planes take the form [78],

$$z = S_0(\zeta) = \alpha_0(t)\zeta + \sum_{n=1}^{\mathcal{D}} \alpha_{-n}(t)\zeta^{-n} \tag{2.6.1a}$$

$$\zeta = S_0^{-1}(z) = \frac{z}{\alpha_0} - \frac{\alpha_{-2}}{z} + \dots \ . \tag{2.6.1b}$$

We now introduce a star $*$ symbol to denote the pull-back of a function in the z-plane to the ζ-plane obtained by substituting (2.6.1a) for z in that function. The pull-backs of $\phi_1(z)$ and $\phi_2(z)$ are denoted by

$$\phi_1^*(\zeta) = \sum_{k<0} a_k^* \zeta^{k+1} \tag{2.6.2a}$$

$$\phi_2^*(\zeta) = \sum_{k<-1} b_k^* \zeta^{k+1} \ , \tag{2.6.2b}$$

where the a_k^* and b_k^* are a new set of coefficients related to the original a_k and b_k through the conformal mapping.

Next we pull back the velocity field onto the unit circle σ in the ζ-plane so that we can apply the no-slip boundary conditions there and determine the pull-back coefficients a_k^* and b_k^* in terms of the $\alpha_i(t)$. On the unit circle σ the velocity expansion takes the form (suppressing the t dependence)

$$v^*(\sigma) = \phi_1^*(\sigma) - \frac{S(\sigma)}{\partial_\sigma S(\sigma)}\overline{\partial_\sigma \phi_1^*(\sigma)} + \overline{\phi_2^*(\sigma)}\ . \tag{2.6.3}$$

The only coefficients we need to determine the translational and rotational motion of the swimmer are a_{-1} and b_{-2}. This is because a_{-1} gives the fluid flow at infinity, so it determines the translational motion of the swimmer, and b_{-2} is related to the torque on the swimmer, so it determines the rotational motion of the swimmer. Using the conformal mapping (2.6.1), the coefficients a_{-1} and b_{-2} can be expressed in terms of the pulled-back coefficients a_k^* and b_k^* as

$$a_{-1} \ = \ a_{-1}^* \tag{2.6.4a}$$

$$b_{-2} \ = \ \alpha_0 b_{-2}^*\ . \tag{2.6.4b}$$

We can solve for the pulled-back coefficients a_k^* and b_k^* in terms of the parameters α_i using (2.6.3), and then use the pulled-back coefficients to solve for a_{-1} and b_{-2}. As in [78] we find

$$a_{-1} \ = \ -\bar{\alpha}_0^{-1}\alpha_{-3}\dot{\bar{\alpha}}_{-2} \tag{2.6.5a}$$

$$b_{-2} \ = \ \bar{\alpha}_0\dot{\alpha}_0 - \alpha_{-2}\dot{\bar{\alpha}}_{-2} - 2\alpha_{-3}\dot{\bar{\alpha}}_{-3}\ . \tag{2.6.5b}$$

To determine the translational part of the gauge potential we note that the coefficient a_{-1} is a constant contribution to the velocity expansion, which means that the fluid velocity at infinity is uniform and non-zero. Following Section 7.5 of Ref. [97], we argue that a finite-size swimmer located near the origin should not be able to induce a non-zero fluid velocity at infinity, and so we make a Galilean transformation to a frame in which the fluid is at rest at infinity and the swimmer moves with a velocity

$$\mathcal{A}_{tr} \equiv V_x + iV_y = -a_{-1}\ , \tag{2.6.6}$$

where $\mathcal{A}_{tr}$ denotes the translational part of the gauge potential (2.2.11).

To determine the rotational part of the gauge potential we attempt to cancel the torque (2.4.13) on the swimmer by having the swimmer rotate at an appropriately chosen angular velocity ω. In the parameterization (2.5.1) of the swimming stroke, having the swimmer rotate at an angular velocity ω amounts to the replacement

$$\alpha_i \to \alpha_{i,rot} = \alpha_i e^{i\omega t} \; . \tag{2.6.7}$$

Under this replacement we find

$$b_{-2} \to b_{-2,rot} = b_{-2} + i\omega\left(|\alpha_0|^2 + |\alpha_{-2}|^2 + 2|\alpha_{-3}|^2\right) \; , \tag{2.6.8}$$

so that the condition that the net torque on the swimmer vanish becomes

$$\eta^e \mathrm{Im}[b_{-2,rot}] - \eta^o \mathrm{Re}[b_{-2,rot}] = 0 \; . \tag{2.6.9}$$

Solving this equation for ω yields the rotational part of the gauge potential

$$\mathcal{A}_{rot} \equiv \omega = \frac{-\mathrm{Im}[b_{-2}] + \frac{\eta^o}{\eta^e}\mathrm{Re}[b_{-2}]}{|\alpha_0|^2 + |\alpha_{-2}|^2 + 2|\alpha_{-3}|^2} \; . \tag{2.6.10}$$

This expression shows that in the presence of odd viscosity the rotational part of the gauge potential picks up a term proportional to $\mathrm{Re}[b_{-2}]$. For the simple swimming stroke (2.5.1), one can verify by explicit computation that

$$\mathrm{Re}[b_{-2}] = \frac{1}{2\pi}\frac{dA(t)}{dt} \; , \tag{2.6.11}$$

which shows that the odd viscosity contribution to the angular velocity of the swimmer is proportional to the rate of change of the area of the swimmer. This conclusion is not limited to swimming strokes which are conformal maps of degree $\mathcal{D} = 2$, but holds for *generic* swimmers bounded by a closed curve without any self-intersections, as we prove in Appendix A.2.

We can use this area relation and the relation $\Gamma(\infty) = -2\pi\mathrm{Im}[b_{-2}]$ for the circulation of the fluid at infinity to rewrite the angular velocity formula in a way which clearly shows the physical meaning of each term. We find

$$\omega = \frac{\Gamma(\infty) + \frac{\eta^o}{\eta^e}\frac{dA(t)}{dt}}{2\pi(|\alpha_0|^2 + |\alpha_{-2}|^2 + 2|\alpha_{-3}|^2)} \; . \tag{2.6.12}$$

As the ratio of η^o/η^e increases, the angular velocity (2.6.10) grows without bound. Therefore we conclude that in a fluid in which the odd viscosity terms completely dominate the stress tensor (i.e. $\eta^o/\eta^e \to \infty$), the condition that the swimmer experience zero net torque must be satisfied by taking $\frac{dA(t)}{dt} = 0$, otherwise the angular velocity of the swimmer would have to be infinite. So a swimmer in a fluid where odd viscosity effects are dominant must have

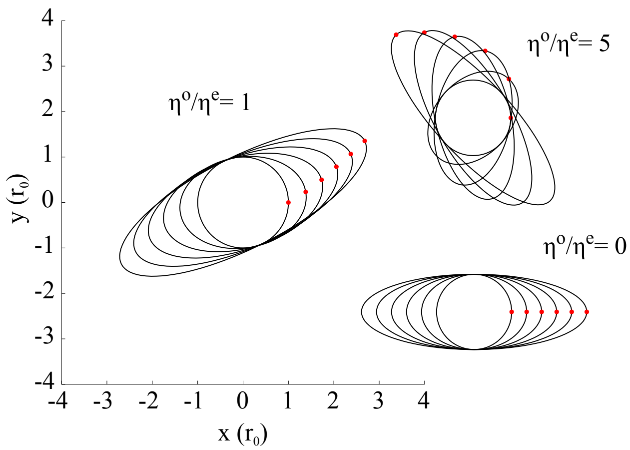

Figure 2.2: The elliptical distortion given by Eq. (2.7.1), shown in three different fluids with different ratios of odd to even viscosity. The time between each consecutive shape is 0.5 units of time. The red dot is a guide for the eye that indicates the same point on the boundary of the shape, so one can clearly see when the shape is rotating and when it is stationary. Distances are measured in units of r_0, which is the original radius of the swimmer before it starts expanding into an ellipse, i.e. $\alpha_0(t = 0) = r_0$.

constant area.

When discussing the limit $\eta^o/\eta^e \to \infty$ in this context, we must always assume that η^e is finite and large enough so that we can still neglect any inertial forces in the problem (and so we can still take advantage of the geometric formulation of the problem of swimming at low Reynolds number). This is why we have been careful to say "when the odd viscosity is dominant" and not "when $\eta^e = 0$."

2.7 Example Swimming Strokes

Here we present some simple examples of swimming strokes that clearly demonstrate the difference between swimming in a fluid with just even viscosity and swimming in a fluid with both even and odd viscosity.

Dipolar Distortion

The first example is a swimmer which starts out as a circle but grows into an ellipse by elongating one of its axes through a dipolar-like distortion. We use the parameterization

$$\alpha_0 = 1 + \tfrac{t}{2} \tag{2.7.1a}$$

$$\alpha_{-2} = \tfrac{t}{2} \tag{2.7.1b}$$

$$\alpha_{-3} = 0 \tag{2.7.1c}$$

for this swimmer. The boundary of the swimmer is an ellipse with the lengths of the major and minor axes given by $a = 1 + t$, $b = 1$. With only the conventional even viscosity this stroke will not cause any motion other than an increase in the area. We can also see this from the reflection symmetry about the x-axis, which is equivalent to the fact that all the coefficients are real. However, when there is also odd viscosity this swimmer will start to rotate because its area is growing and the torque has a term proportional to the odd viscosity and the rate of area change. The motion for different values of the odd viscosity can be seen in Fig. 3.1.

Quadrupolar Distortion

To further test our results we chose a swimmer with a more complicated quadrupolar distortion which also has a uniform area growth. We used the parameterization

$$\alpha_0 = 1 + t \tag{2.7.2a}$$

$$\alpha_{-2} = 0 \tag{2.7.2b}$$

$$\alpha_{-3} = 0 \tag{2.7.2c}$$

$$\alpha_{-4} = \tfrac{1}{4}\,. \tag{2.7.2d}$$

This swimming parameterization represents a conformal map of degree $\mathcal{D} = 3$. To see how to extend the analysis of the previous section to swimmers which are conformal maps of the circle of degree $\mathcal{D} = 3$ (i.e. how to include α_{-4} terms), see Appendix A.3. In Fig. 2.3 we see very similar results to the dipolar case, e.g., the motion of the swimmer is just a rotation proportional to the growth of the area. This indicates, as we expected from the general result of Appendix A.2, that the odd viscosity does not distinguish between different types of shape distortions, and only couples to changes in the total area of the interior of the swimmer.

Wandering Stroke

The third example is a swimmer parameterized with the cyclic stroke

$$\alpha_0 = r_0 \tag{2.7.3a}$$

$$\alpha_{-2} = -i\xi_1 \sin(2\pi t) \tag{2.7.3b}$$

$$\alpha_{-3} = -i\xi_2 \cos(2\pi t) \tag{2.7.3c}$$

where r_0, ξ_1 and ξ_2 are all real parameters. We chose this particular stroke because in the case when only the even viscosity is present, the swimmer's centroid moves in a straight line through the fluid. Additionally, this stroke has a

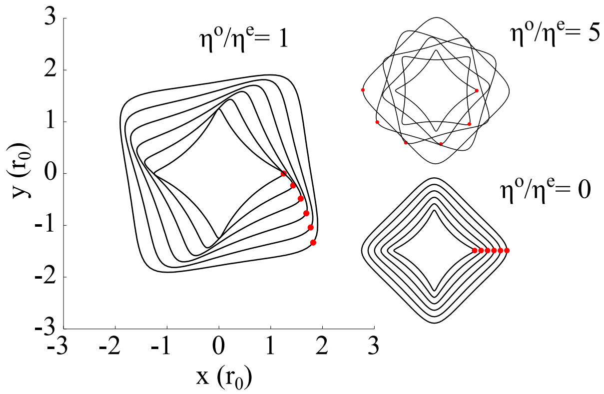

Figure 2.3: The quadrupolar distortion given by Eq. (2.7.2), shown in three different fluids with different ratios of odd to even viscosity. The time between each consecutive shape is 0.5 units of time. The red dot is a guide for the eye that indicates the same point on the boundary of the shape, so one can clearly see when the shape is rotating and when it is not. Distances are measured in units of r_0, defined by the relation $\alpha_0(t=0) = r_0$.

periodic time-dependent area

$$A(t) = \pi[r_0^2 - \xi_1^2 + (\xi_1^2 - 2\xi_2^2)\cos^2(2\pi t)] , \tag{2.7.4}$$

which implies that it will feel a cyclic stress from the odd viscosity term when present. In Fig. 2.4 one can clearly see the outcome as we show two trajectories, one with $\eta^o/\eta^e = 0$ and one with $\eta^o/\eta^e = 10$. In the case when η^o vanishes, the swimmer travels in a straight line, however in the second case the swimmer oscillates transverse to the straight-line path. In the inset we show that the amplitude of the transverse oscillation at a fixed time increases linearly with the slope η^o/η^e. As the swimmer continues it will wander further and further off of the straight-line course although on average it seems like it will still progress linearly at a similar rate to that of the swimmer in the fluid with vanishing odd viscosity.

Null-Rotation Stroke

The fourth example is a stroke which will nominally rotate when just even viscosity is present, but for which the variation of the area of the shape has been chosen carefully so that when odd viscosity is also present the shape will not rotate at all. In other words, the odd viscosity contribution to the angular velocity exactly cancels the even viscosity contribution for a given particular ratio η^o/η^e which, for the sake of this example, we pick to be unity.

A glance at Eq. (2.6.10) shows that in order to produce this cancellation, we need the stroke to satisfy

$$\mathrm{Im}[b_{-2}] = \mathrm{Re}[b_{-2}] . \tag{2.7.5}$$

For swimmers which are conformal maps of the circle of degree $\mathcal{D} = 2$, the coefficient b_{-2} is given by Eq. (2.6.5b).

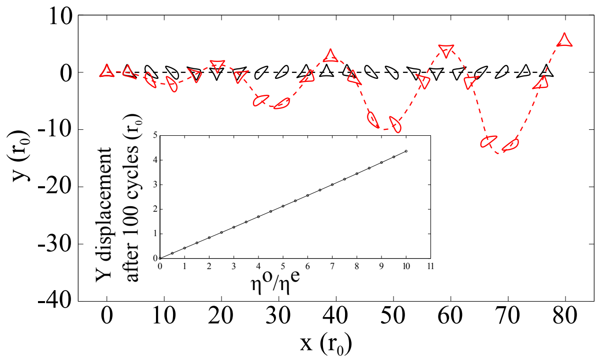

Figure 2.4: The swimming stroke of Eq. (2.7.3) with the parameter values $r_0 = 1$, $\xi_1 = 0.5$, and $\xi_2 = 0.4$, shown first with just even viscosity (in black) and then with both even and odd viscosity (in red) with $\eta^o/\eta^e = 10$. The time between each consecutive shape in the figure is 6.1 cycles. When odd viscosity is also present, the swimmer wanders off of its straight trajectory because of rotations caused by changes in the area of the swimmer. The inset shows the y-displacement of the swimmer after 100 cycles of this swimming stroke vs. the ratio of the odd and even viscosity coefficients. Distances are measured in units of r_0, the parameter that appears in Eq. 2.7.3a.

We see from that equation that we can design such a stroke by taking $\alpha_0 = r_0 =$ constant and

$$\alpha_j(t) = r_j(t)e^{i\theta_j(t)} \tag{2.7.6}$$

for $j = -2, -3$, where the functions $r_j(t)$ and $\theta_j(t)$ are functions which are determined in the following way. We would like to have

$$\alpha_j(t)\dot{\bar{\alpha}}_j(t) = (1+i)\dot{f}_j(t) \tag{2.7.7}$$

where the $f_j(t)$ are some real periodic functions of time (to give a periodic swimming stroke), which we are essentially free to choose. This choice will guarantee the cancellation of the even and odd viscosity contributions to the torque on the swimmer, since the real and imaginary parts of Eq. (2.7.7) are equal at all times. The reason for using the derivative of the functions $f_j(t)$ in the above formula is purely for convenience in the formulas that follow. Plugging the form (2.7.6) for the $\alpha_j(t)$ into this last equation and solving the two coupled ordinary differential equations for $r_j(t)$ and $\theta_j(t)$ gives the form of the stroke in terms of the functions $f_j(t)$,

$$r_j(t) = \sqrt{2\big(f_j(t) + C_{j,1}\big)} \tag{2.7.8a}$$

$$\theta_j(t) = -\frac{1}{2}\ln\big(f_j(t) + C_{j,1}\big) + C_{j,2}\,, \tag{2.7.8b}$$

where $C_{j,1}$ and $C_{j,2}$ are arbitrary constants (although they must be chosen carefully along with the functions f_j to keep the argument of the logarithm from ever equaling zero). Now any choice of the periodic functions $f_j(t)$ will give a cyclic swimming stroke that will not rotate in a fluid with our chosen ratio $\eta^o/\eta^e = 1$.

This shows in principle that it is possible to construct a stroke for which the even and odd viscosity contributions to the angular velocity exactly cancel each other. Swimmers using this type of stroke might be able to more efficiently navigate odd-viscosity fluids since the particular choice of stroke cancels the rotation effects due to the odd-viscosity.

2.8 Reciprocal Motions and Scallop Theorem with Odd Viscosity

An interesting aspect of swimming at low Reynolds number in an ordinary viscous fluid is the fact that a reciprocal swimming stroke leads to no net motion of the swimmer through the fluid. By a reciprocal swimming stroke we mean a swimming stroke which looks exactly the same whether time is run forwards or backwards. This fact has become known as the Scallop theorem [24]. The opening and closing of a scallop's shell is the prototypical example of a reciprocal stroke. Examples of non-reciprocal strokes include corkscrew and undulatory motions. Reversing time in those situations will reverse the direction of rotation of the corkscrew motion, and it will reverse the direction of travel of the waves in the undulatory motion.

One can understand this result using the uniqueness theorem for the solutions of the slow flow equations, as proved in Ref. [97]. Running time backwards corresponds to negating the velocity of the fluid at the surface of the swimmer. In other words, the boundary condition for the time-reversed flow is obtained by changing the sign of $\mathbf{v}$ in the no-slip boundary conditions at the surface of the swimmer.

With only even viscosity the unique solution to the slow flow equations with time-reversed boundary conditions is the time-reverse of the solution with the original boundary conditions (i.e. the velocity is negated everywhere). This means that whatever motion the scallop does as it opens its shell is immediately undone when it closes its shell. Therefore the scallop can make no net progress.

In the geometric theory of swimming at low Reynolds number one can make sense of this result by noting that a reciprocal swimming motion encloses no area in the space of un-located shapes, therefore the reverse path-ordered integral of Eq. (2.2.5) is just the identity matrix.

Now we ask whether this result changes when we include odd viscosity. When the effects of odd viscosity are included there is the additional possibility that the swimmer can rotate itself with the reciprocal motion, and that the interplay between rotations and translations could lead to a net displacement after a full cycle of the swimming stroke, even through the stroke is reciprocal.

In fact, this is not the case, and the Scallop theorem still holds in fluids with both even and odd viscosity. The uniqueness theorem argument is still valid in this case. One just needs to prove that the slow flow equations with odd viscosity terms still have unique solutions. In appendix A.4 we extend the usual uniqueness proof for the slow flow

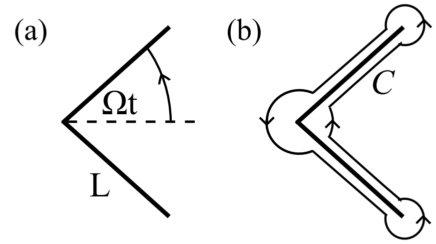

Figure 2.5: (a) The simple model for the scallop swimmer. The legs have a length L and the scallop opens symmetrically about the x-axis with angular velocity Ω. (b) The contour C we use to evaluate the flux of the scallop swimming stroke. This contour hugs the scallop tightly but circles around the wedge-shaped region in the center and also around the ends of the two arms.

equations with just even viscosity (see Ref. [97]) to the case where odd viscosity terms are present. There we also give a separate uniqueness argument which applies even to swimmers whose boundary is not a smooth closed curve (for example the simple model of the scallop in Fig. 2.5a that we consider later in this section). Therefore we can conclude that the Scallop theorem is also true in viscous fluids with both even and odd viscosity.

2.8.1 No Flux for the Scallop Swimming Stroke

In the remainder of this section we give an argument showing that for a simple model of the scallop swimming stroke the flux of the fluid at infinity vanishes. This means that the scallop cannot rotate at all in a fluid with odd viscosity. Note that this is a stronger statement than the Scallop theorem, which only states that a reciprocal swimming stroke cannot give any *net* motion (translation or rotation) through the fluid after one cycle.

We model the scallop as two infinitely thin arms of length L connected at the origin. Let Ωt be the angle between each arm and the positive x-axis, so Ω is the angular velocity of the stroke at time t (see Fig. 2.5a). The no slip boundary conditions for the fluid on the scallop are then

$$\mathbf{v}(r, \pm\Omega t) = r\Omega(\mp\sin(\Omega t)\hat{\mathbf{x}} + \cos(\Omega t)\hat{\mathbf{y}}),\ r < L\ . \tag{2.8.1}$$

Now we want to evaluate $\Phi(\infty)$, the flux of the fluid at infinity, due to the scallop swimming stroke. Since the fluid is incompressible, we can calculate this flux with any contour we want, instead of using the circular contour at infinity. We choose a contour C shown in Fig. 2.5b. This contour hugs the scallop tightly but circles around the wedge-shaped region near the origin and around the ends of the arms.

The contribution to the flux from the straight parts of the contour vanishes, since on one side of each arm the velocity points towards the inwards normal and on the other side it points towards the outwards normal. So we just have to worry about the contribution to the flux from the small circular parts of the contour which surround the hinge of the scallop and the two ends. Therefore we need solutions to the viscous equations of motion which are valid in these small regions. We argue that we can use solutions to the equations for infinite geometries, since those should be approximately correct when we are very close to these small regions. Writing $\mathbf{v} = \nabla \times (\psi \hat{\mathbf{z}})$, where ψ is the stream function, the basic equation we need to solve is the biharmonic equation for ψ,

$$\nabla^2(\nabla^2\psi) = 0 \, . \tag{2.8.2}$$

Let us suppose that $\Omega t < \pi/2$ and look at the flux from each small circular contour on either side of the wedge. We take the contour to have radius ϵ and the angle θ for integration over the contour ranges from $-\Omega t + \delta \leq \theta \leq \Omega t - \delta$ for the inner part of the wedge and from $\Omega t + \delta \leq \theta \leq 2\pi - \Omega t - \delta$ for the outer part of the wedge, where δ is a very small angle. For this infinite wedge geometry (i.e. $L \to \infty$), Eq. 2.8.2 has the solution (see Ref. [97])

$$\psi(r,\theta) = -\frac{1}{2}\Omega r^2 \left(\frac{\sin(2\theta) - 2\theta\cos(2\Omega t)}{\sin(2\Omega t) - 2\Omega t \cos(2\Omega t)} \right) \, . \tag{2.8.3}$$

This solution is valid for $2\Omega t \lesssim 257.45°$, which is the angle where the denominator equals zero. Near that angle $\psi(r,\theta)$ shows more complicated scaling behavior and exhibits several different scaling regimes. Below, but very near, the critical angle $\psi(r,\theta)$ will instead scale with r as r^{p_2+2} and then deform to scale as $r^2\ln(r)$. For $2\Omega t$ greater than the critical angle $\psi(r,\theta)$ has three different regimes and will scale as r^{p_1+2}, r^2 and then r^{p_3+2}, where p_1, p_2 and p_3 are complex numbers with $\text{Re}[p_i] > -1, \; i = 1,2,3$, though p_1 is actually real. The detailed solution to the infinite geometry wedge problem for all angles Ωt is discussed in Ref. [98], though we do not need much of the detail for what we are studying.

Now the radial and angular components of the fluid velocity are given in terms of the stream function by

$$v_r = \frac{1}{r}\frac{\partial\psi}{\partial\theta}, \; v_\theta = -\frac{\partial\psi}{\partial r} \, . \tag{2.8.4}$$

The radial component is relevant for the computation of the fluxes

$$\Phi_{wedge,1} = \int_{-\Omega t+\delta}^{\Omega t-\delta} v_r(\epsilon,\theta)\epsilon d\theta \tag{2.8.5}$$

$$\Phi_{wedge,2} = \int_{\Omega t+\delta}^{2\pi-\Omega t-\delta} v_r(\epsilon,\theta)\epsilon d\theta \tag{2.8.6}$$

where ϵ is again the radius of the circular contour. We see that for all angles Ωt, the product $\epsilon v_r(\epsilon,\theta)$ scales as $\epsilon^\alpha, \; \text{Re}[\alpha] > 1$, possibly multiplied by $\ln(\epsilon)$, which means that when we take the limit $\epsilon \to 0$ (the circle shrinks to zero radius), these contributions to the flux will vanish.

Next we have to look at the flow near the very ends of the arms. We argue that the flow here can be approximated by the flow near the end of a semi-infinite line or plate being dragged through the fluid with a velocity perpendicular to it's length. For simplicity, we look at the solution of the slow flow equations where the semi-infinite plate occupies the negative x-axis and is moving in the negative y direction with speed v_0. We again want to solve Eq. 2.8.2, but now subject to the boundary conditions

$$\mathbf{v}(r,\pi) = v_0\hat{\boldsymbol{\theta}}\ . \tag{2.8.7}$$

This time we find that the solution is

$$\psi(r,\theta) = r(A\cos(\theta) + B\sin(\theta) + C\theta\cos(\theta) + D\theta\sin(\theta))\ , \tag{2.8.8}$$

where A, B, C, D are constants that must be determined from the boundary conditions. Imposing the no-slip boundary conditions at $\theta = \pi$ gives

$$C = \frac{v_0 - A}{\pi} \tag{2.8.9}$$

$$D = -\frac{B}{\pi} + \frac{A - v_0}{\pi^2}\ . \tag{2.8.10}$$

The important feature of this solution is that the fluid velocity actually scales as r^0. Now the semi-infinite plate problem is unrealistic, and this is reflected in the fact that in the solution the velocity has no r dependence at all. However, we expect this solution to still be valid very close to the tip of the plate, and that is where we will make use of it. We evaluate the flux of the fluid around a circular contour of radius ϵ with center at the origin (end of the plate) and $-\pi + \delta \leq \theta \leq \pi - \delta$. Since the fluid velocity scales there as r^0, the factor of ϵ we get from the line element $ds = \epsilon d\theta$ in the integration is the only factor of ϵ present, and so this contribution to the flux vanishes as we take $\epsilon \to 0$. Therefore we conclude that the contribution to the total flux from the two ends of the scallop is also zero, and so the total flux $\phi(\infty)$ of the scallop swimming stroke is zero. This means that in an odd viscosity fluid the scallop cannot rotate at all as it opens and closes its shell.

2.9 Conclusion

We have applied the geometric theory of swimming at low Reynolds number developed by Wilczek and Shapere [78] to the case where the fluid has a non-vanishing odd, or Hall, viscosity. The main effect of the odd viscosity is to introduce an additional torque on the swimmer, proportional to the rate of change of the area of the swimmer, independent of the other shape changes occurring in the stroke pattern. This torque is the companion effect to the fact that a swimmer rotating in a fluid with odd viscosity feels an inwards or outwards pressure proportional to its angular velocity [23]. As we show in Appendix A.2 this conclusion applies to generic swimming shapes and is not limited to swimmers whose boundaries are simple conformal maps of the unit circle.

As a consequence of this extra torque, a swimming stroke which would not cause the swimmer to rotate in a fluid with conventional viscosity can cause the swimmer to rotate in a fluid with odd viscosity if the area of the swimmer is changing. It is even possible to design a stroke which will rotate the swimmer in an even viscosity fluid but not in a fluid with both even and odd viscosity, for a certain value of the ratio η^o/η^e. It is possible that swimmers placed in fluids with an odd viscosity would have to adapt their strokes to efficiently move in a straight line. Additionally, it would be interesting to see if swimmers could use the interplay between the even and odd viscosity to perform more interesting or efficient motion patterns.

Chapter 3

Hall viscosity and geometric response in the Chern-Simons matrix model of the Laughlin states[1]

3.1 Introduction

In the past few years there has been tremendous progress in the understanding of the geometric properties of quantum Hall states. The role of geometry in the quantum Hall effect first came to prominence in early work on Hall viscosity [22, 23, 86] (sometimes called *odd* viscosity), and there has been much work on Hall viscosity since then [87–93, 99–101]. Recent work on geometry in the quantum Hall effect has gone in two separate directions. First, there is the study of the properties of quantum Hall states on curved spatial manifolds (Riemann surfaces) [102–109]. Second, there is the study of intrinsic geometry and anisotropy in quantum Hall states on flat space [89, 100, 101, 110, 111]. In the past year a very interesting new theory of quantum Hall states has been proposed, known as the *bi-metric* theory, and this theory promises to unify the two separate directions of research on geometry in the quantum Hall effect [112, 113].

In a separate line of development, Susskind proposed in 2001 that Laughlin fractional quantum Hall (FQH) states could be described by *noncommutative Chern-Simons* (NCCS) theory [27]. This is a deformation of ordinary Chern-Simons theory in which the theory is formulated on a noncommutative analog of the flat space $\mathbb{R}^2$ consisting of "coordinates" $\hat{x}^1$ and $\hat{x}^2$ obeying a nontrivial commutation relation

$$[\hat{x}^1, \hat{x}^2] = i\theta \ . \tag{3.1.1}$$

Here θ is a real parameter with dimensions of length squared that characterizes the degree of noncommutativity of the theory. The original motivation for this proposal was Susskind's observation that the gauge symmetry of NCCS theory provides a discretization of the symmetry under area-preserving diffeomorphisms that is present in a description of a FQH state as a charged fluid in a magnetic field. In particular, this discretization was argued to capture the "granularity" of a fluid constructed from electrons, and in the NCCS theory description each electron is associated

[1] This Chapter is based on the arXiv preprint 1802.10100 which is cited as Ref. [30] in the References section of the thesis.

with a fundamental area of size $2\pi|\theta|$. In addition, in the NCCS theory a quantization rule [114] enforces

$$\theta = \ell_B^2 m \ , \ m \in \mathbb{Z} \ , \tag{3.1.2}$$

where ℓ_B is the magnetic length, and so one finds (for $m > 0$) that the fluid described by the NCCS theory has a number density $\rho_0 = \frac{1}{2\pi\ell_B^2 m}$, exactly as in the $\nu = \frac{1}{m}$ Laughlin state.

Susskind's original proposal has been followed up by many authors [29, 115–123]. Of all of these subsequent works, the work of Polychronakos is particularly important for this Chapter. In Ref. [29], Polychronakos proposed a regularization of the NCCS theory which is known as the *Chern-Simons matrix model* (CSMM). This is a particular regularization of the NCCS theory which can be viewed as a quantum mechanics model in which the degrees of freedom are $N \times N$ matrices (we discuss the model in detail and make this statement precise below). The quantum ground state of the CSMM having $\theta = \ell_B^2 m$ ($m > 0$) is known to describe a uniform droplet of "noncommutative fluid" with a density and area matching that of the $\nu = \frac{1}{m}$ Laughlin state. Polychronakos has also demonstrated that excitations in this model carry the appropriate fractional charge of the quasihole excitations in the Laughlin state.

Despite the successes in describing the basic properties of the Laughlin FQH states using NCCS theory and the CSMM, there have not been any attempts to study *geometric* properties of the Laughlin states in the context of these noncommutative models. Therefore, our goal in this Chapter is to answer the following question: *does the CSMM accurately describe the geometric properties of the Laughlin states?*

The particular geometric properties that we are concerned with are the Hall viscosity, the Hall conductance in a non-uniform electric field, and the Hall viscosity in the presence of anisotropy (or intrinsic geometry). We compute all of these quantities in the CSMM and we find that the results in the CSMM contain only the *guiding center* contribution to the known values for these quantities in the Laughlin states. For example, the full Hall viscosity coefficient for the Laughlin $\nu = \frac{1}{m}$ state is given by [87]

$$\eta_{tot} = \frac{\hbar\rho_0 m}{4} \ , \tag{3.1.3}$$

while for the CSMM with $\theta = \ell_B^2 m$ we find[2] (after regularization)

$$\eta_{\text{CSMM},reg} = \frac{1}{2}\hbar\rho_0\left(\frac{m-1}{2}\right) \ , \tag{3.1.4}$$

which is exactly the (regularized) *guiding center Hall viscosity* of the $\nu = \frac{1}{m}$ Laughlin state [89, 100, 101]. The need for regularization of the guiding center part of the Hall viscosity has been discussed in Refs. [89, 100, 101]. In this Chapter we also give a fluid interpretation of this regularization in the context of the CSMM.

[2]In the literature the quantity $\frac{m-1}{2}$ is referred to either as the *anisospin* (Refs. [112, 113]) or minus the *guiding center spin* (Refs. [89, 100, 101]) of the $\nu = \frac{1}{m}$ Laughlin state.

Based on our calculations we conclude quite generally that the CSMM and NCCS theory descriptions of the Laughlin FQH states capture the guiding center contribution to the geometric properties of these states, but lack the Landau orbit contribution. We argue that this is not surprising since in the fluid interpretation of the CSMM and NCCS theories, the cyclotron frequency ω_c is sent to infinity by sending the mass of the particles in the fluid to zero. This is analogous to a projection into a Landau level (which freezes out the Landau orbit degrees of freedom), and so it makes sense that only the guiding center contribution remains. The Landau orbit contribution is often considered to be less important since the interesting correlations in a Laughlin state are contained entirely in the guiding center part of the state/wave function. Therefore we find that the CSMM description is able to capture the most important contributions to the geometric properties of the Laughlin states. We expect that our results will rekindle interest in noncommutative models of the FQH effect, as these models clearly have a role to play in the investigation of geometric properties of FQH states.

This Chapter is organized as follows. In Sec. 3.2 we review the notion of Hall viscosity. In Secs. 3.3 and 3.4 we give a comprehensive review of the NCCS theory and CSMM, the fluid interpretation of these models, and their relation to the Laughlin states. In Sec. 3.5 we compute the Hall viscosity in the CSMM. In Sec. 3.6 we compute the Hall conductance of the CSMM in a non-uniform electric field. In Sec. 3.7 we give a fluid interpretation of the regularization of the guiding center part of the Hall viscosity in which one subtracts the extensive contribution to this quantity. Finally, in Sec. 3.8 we present a modified version of the CSMM incorporating anisotropy, and we compute the Hall viscosity for the modified model. Sec. 3.9 presents our conclusions. This Chapter also includes two appendices. In Appendix B.1 we review the form of the quantum generators of the action of the group $U(N)$ on the fields of the CSMM, as this information is necessary for the quantization of this model which we review in Sec. 3.4. In Appendix B.2 we present the details of the calculation of the Hall viscosity of the CSMM (which is presented in Sec. 3.5 of the main text), which involves a Kubo formula approach inspired by Ref. [92].

3.2 Review of Hall viscosity

In this section we review the concept of Hall viscosity following the derivation and point of view in Ref. [101]. We also emphasize, again following Ref. [101], the separation of the Hall viscosity tensor into two parts: the *Landau orbit* contribution and the *guiding center* contribution. Finally, we review the form of both parts of the Hall viscosity tensor for typical FQH trial states including the Laughlin states. The example of the Laughlin states is of particular interest for the rest of the Chapter when we compare to the results obtained in the CSMM, which has been argued to describe the physics of the Laughlin states.

3.2.1 Hall viscosity calculation

The Hall viscosity can be computed by studying the response of a FQH state to time-dependent area-preserving deformations (APDs). Before we review the calculation of the Hall viscosity, we briefly recall the setup of the quantum Hall problem. We consider N electrons on the plane, each with a charge $-e < 0$, in the presence of a constant background magnetic field of strength $B > 0$ and pointing in the positive z direction. Let $\mathbf{r}_j$ be the position coordinates of the N electrons, where $j = 1, \dots, N$, is a particle label. We write r_j^a with $a = 1, 2$, for the two components of the vector $\mathbf{r}_j$ (i.e., $a = 1, 2$, labels the two directions of space). In this situation the electron coordinate operators r_j^a break up into two parts as

$$r_j^a = R_j^a + \tilde{R}_j^a \ , \tag{3.2.1}$$

where R_j^a are known as the guiding center coordinates, and $\tilde{R}_j^a$ are the Landau orbit coordinates. These coordinates obey the commutation relations

$$\left[R_j^a, R_k^b\right] = i\ell_B^2 \epsilon^{ab} \delta_{jk} \tag{3.2.2a}$$

$$\left[\tilde{R}_j^a, \tilde{R}_k^b\right] = -i\ell_B^2 \epsilon^{ab} \delta_{jk} \tag{3.2.2b}$$

$$\left[R_j^a, \tilde{R}_k^b\right] = 0 \ , \tag{3.2.2c}$$

where $\ell_B^2 = \frac{\hbar}{eB}$ is the square of the magnetic length ℓ_B.

The Hall viscosity is defined as the response of the system (more precisely, the ground state) to time-dependent, APDs of the electron coordinates. These APDs are generated by Hermitian operators $\mathsf{\Lambda}^{ab}$ which are a linear combination of guiding center and Landau orbit parts,

$$\mathsf{\Lambda}^{ab} = \Lambda^{ab} - \tilde{\Lambda}^{ab} \ . \tag{3.2.3}$$

The operators Λ^{ab} generate APDs of the guiding center coordinates and have the form

$$\Lambda^{ab} = \frac{1}{4\ell_B^2} \sum_{j=1}^{N} \left\{R_j^a, R_j^b\right\} \ , \tag{3.2.4}$$

where $\{\cdot, \cdot\}$ denotes an anti-commutator, while $\tilde{\Lambda}^{ab}$ generates APDs of the Landau orbit coordinates, and $\tilde{\Lambda}^{ab}$ is defined like Λ^{ab} but with the guiding center coordinates R_j^a replaced by the Landau orbit coordinates $\tilde{R}_j^a$. One can show that these generators obey the Lie algebras

$$[\Lambda^{ab}, \Lambda^{cd}] = \frac{i}{2}\left(\epsilon^{bc}\Lambda^{ad} + \epsilon^{bd}\Lambda^{ac} + \epsilon^{ac}\Lambda^{bd} + \epsilon^{ad}\Lambda^{bc}\right) \tag{3.2.5a}$$

$$\left[\tilde{\Lambda}^{ab}, \tilde{\Lambda}^{cd}\right] = -\frac{i}{2}\left(\epsilon^{bc}\tilde{\Lambda}^{ad} + \epsilon^{bd}\tilde{\Lambda}^{ac} + \epsilon^{ac}\tilde{\Lambda}^{bd} + \epsilon^{ad}\tilde{\Lambda}^{bc}\right) \ . \tag{3.2.5b}$$

In addition, it is clear that $[\Lambda^{ab}, \tilde{\Lambda}^{cd}] = 0$. The generators Λ^{ab} (and also $\tilde{\Lambda}^{ab}$) can be expressed in terms of the generators of the Lie algebra of the group $SU(1,1)$, and we will use this fact later[3].

Finite (as opposed to infinitesimal) APDs of the electron coordinates are implemented by conjugation by the unitary operators[4]

$$U(\alpha) = e^{i\alpha_{ab}\Lambda^{ab}} \ , \tag{3.2.6}$$

where α_{ab} is a constant, symmetric tensor with unit determinant (thus, the APDs are spatially uniform since α_{ab} does not depend on the spatial coordinates). For example, acting on the electron coordinates gives

$$U(\alpha) r^a_j U(\alpha)^\dagger = r^a_j + \epsilon^{ab}\alpha_{bc} r^c_j + \dots \ , \tag{3.2.7}$$

where the ellipses denote higher order terms in α_{ab}.

The APDs that we have been considering so far are closely related to strains in continuum mechanics. Suppose the vector $\mathbf{r}$ is the location of a point in a solid or fluid before a deformation, and $\mathbf{r}'(\mathbf{r})$ is the location of that same point after the deformation. Then for small deformations the *strain tensor* u_{ab} is defined in terms of the displacement vector $\mathbf{u}(\mathbf{r}) = \mathbf{r}'(\mathbf{r}) - \mathbf{r}$ as

$$u_{ab} = \frac{1}{2}\left(\frac{\partial u_a}{\partial r^b} + \frac{\partial u_b}{\partial r^a}\right) \ . \tag{3.2.8}$$

If we consider small APDs in the quantum Hall problem (i.e., we work to linear order in α_{ab}), then we find a strain tensor

$$u_{ab} = \frac{1}{2}\left(\delta_{ac}\epsilon^{cd}\alpha_{db} + (a \leftrightarrow b)\right) \ . \tag{3.2.9}$$

In particular, we find for the trace $\sum_{a=1}^{2} u_{aa} = 0$, which means that the APDs are indeed area-preserving (the trace of the strain tensor determines the change in the area of a small element of the fluid or solid at the location $\mathbf{r}$). The strain tensor is also spatially uniform since α_{ab} does not depend on the spatial coordinates r^a. Therefore, the APDs that we have been considering can be understood as a special case of a strain in continuum mechanics, namely, a spatially uniform and area-preserving strain. In what follows we sometimes use the terms APD and strain interchangeably

[3]Physicists can read about the group $SU(1,1)$ in Ref. [124], for example

[4]Here, and in the rest of the Chapter, we use a summation convention in which we sum over all indices which are repeated once as a subscript and once as a superscript. All other summations will be indicated explicitly.

although, strictly speaking, the former is a special case of the latter.

Consider a FQH system described by a Hamiltonian H_0. Under a time-independent APD parametrized by α_{ab} the Hamiltonian is transformed to

$$H(\alpha) = U(\alpha)H_0U(\alpha)^\dagger \ . \tag{3.2.10}$$

We can define the generalized force associated with this APD as

$$F^{ab} = -\frac{\partial H(\alpha)}{\partial \alpha_{ab}}\Big|_{\alpha=0} = -i[\Lambda^{ab}, H_0] \ . \tag{3.2.11}$$

If we subject the system to a time-dependent APD $\alpha_{ab}(t)$, then we can compute the expectation value of F^{ab} in the time-dependent ground state $|\psi(t)\rangle$ in an expansion in time derivatives of $\alpha_{ab}(t)$. In fact, as was argued in Ref. [92], one should actually compute the expectation value of $U(\alpha(t))F^{ab}U(\alpha(t))^\dagger$ instead of F^{ab}. We discuss this point in more detail in the context of our Kubo formula calculation of the Hall viscosity for the CSMM in Appendix B.2, but just mention here that this replacement corresponds to expressing the generalized force in terms of the coordinates of the deformed system.

We now compute the expectation value of $U(\alpha(t))F^{ab}U(\alpha(t))^\dagger$ in an expansion in time derivatives of $\alpha_{ab}(t)$ as

$$\langle\psi(t)|U(\alpha(t))F^{ab}U(\alpha(t))^\dagger|\psi(t)\rangle = \langle\psi_0|F^{ab}|\psi_0\rangle + \Gamma^{abcd}\dot{\alpha}_{cd}(t) + \dots \ , \tag{3.2.12}$$

where $|\psi_0\rangle$ denotes the initial state of the system, the overdot on $\alpha_{cd}(t)$ denotes a time derivative, and Γ^{abcd} is a four index tensor which is independent of the parameters $\alpha_{ab}(t)$ (in principle there could also be an elastic term which is proportional to $\alpha_{ab}(t)$, but this term is not present for a fluid state). Park and Haldane then define the full Hall viscosity tensor η^{abcd}_{tot} (with all indices raised) as

$$\eta^{abcd}_{tot} = \frac{\Gamma^{abcd}}{A} \ , \tag{3.2.13}$$

where A denotes the area of the quantum Hall droplet (recall that we are working on the infinite plane, so we must assume that the quantum Hall droplet occupies a finite area A). The intuition behind this definition is that η^{abcd}_{tot} encodes the linear response of the "generalized stress" $\frac{U(\alpha(t))F^{ab}U(\alpha(t))^\dagger}{A}$ to the "rate of strain" encoded by the tensor $\dot{\alpha}_{cd}(t)$. We also note here that for a droplet of quantum Hall fluid the area A of the droplet can be expressed as $A = 2\pi\ell_B^2 N_\phi$, where N_ϕ is the number of fundamental flux quanta $\Phi_0 = \frac{h}{e}$ piercing the droplet.

Using adiabatic perturbation theory, Park and Haldane showed that

$$\begin{aligned}\eta^{abcd}_{tot} &= \frac{i\hbar}{A}\langle\psi_0|[\Lambda^{ab},\Lambda^{cd}]|\psi_0\rangle \\ &= \frac{i\hbar}{A}\langle\psi_0|[\Lambda^{ab},\Lambda^{cd}]|\psi_0\rangle + \frac{i\hbar}{A}\langle\psi_0|[\tilde{\Lambda}^{ab},\tilde{\Lambda}^{cd}]|\psi_0\rangle \\ &\equiv \eta^{abcd}_H + \tilde{\eta}^{abcd}_H\ . \end{aligned} \tag{3.2.14}$$

Thus, the full Hall viscosity tensor breaks up into two parts: the guiding center Hall viscosity tensor η^{abcd}_H, and the Landau orbit Hall viscosity tensor $\tilde{\eta}^{abcd}_H$.

The expression for the full Hall viscosity tensor can be simplified further by using the algebra of APD generators from Eq. (3.2.5) to find

$$\eta^{abcd}_{tot} = \frac{1}{2}\left(\epsilon^{ac}\eta^{bd}_{tot} + \epsilon^{ad}\eta^{bc}_{tot} + (a\leftrightarrow b)\right)\ , \tag{3.2.15}$$

where the symmetric two-index tensor η^{ab}_{tot} also breaks up into guiding center and Landau orbit parts as

$$\eta^{ab}_{tot} = \eta^{ab}_H + \tilde{\eta}^{ab}_H \tag{3.2.16}$$

with

$$\eta^{ab}_H = -\frac{\hbar}{A}\langle\psi_0|\Lambda^{ab}|\psi_0\rangle \tag{3.2.17a}$$

$$\tilde{\eta}^{ab}_H = \frac{\hbar}{A}\langle\psi_0|\tilde{\Lambda}^{ab}|\psi_0\rangle\ . \tag{3.2.17b}$$

In what follows we also refer to these two-index tensors as "Hall viscosity tensors". Ref. [101] emphasized that the guiding center contribution η^{ab}_H to η^{ab}_{tot} has a physical interpretation in terms of the intrinsic electric dipole moment along the edge of a FQH state, and in fact must be proportional to the symmetric tensor which determines this dipole moment in order to balance the force on a FQH edge in an inhomogeneous electric field (see also Ref. [125] for a complementary discussion of this boundary dipole moment from a different point of view). We now review the form of the two parts of the Hall viscosity tensor for typical FQH trial states including the Laughlin states.

3.2.2 Values in quantum Hall trial states

In this section we consider the form of the guiding center and Landau orbit Hall viscosity tensors η^{ab}_H and $\tilde{\eta}^{ab}_H$ for typical FQH trial states including the Laughlin states. In the operator, or Heisenberg, approach (as opposed to the Schrodinger approach using wave functions) a state vector for a trial FQH state is constructed using ladder operators

b_j and $b_j^\dagger$ defined in terms of the guiding center coordinates as

$$b_j = \frac{1}{\ell_B\sqrt{2}}(R_j^1 + iR_j^2)\ , \tag{3.2.18}$$

and also ladder operators a_j and $a_j^\dagger$ defined in terms of the Landau orbit coordinates as

$$a_j = \frac{1}{\ell_B\sqrt{2}}(\tilde{R}_j^1 - i\tilde{R}_j^2)\ . \tag{3.2.19}$$

We define $|0\rangle_a$ and $|0\rangle_b$ to be the Fock vacuum states annihilated by the a_j and b_j operators, respectively. In terms of these, a typical FQH trial state in the n^{th} Landau level has the form

$$|\psi_0\rangle = \left[\prod_{j=1}^{N}\frac{(a_j^\dagger)^n}{\sqrt{n!}}\right] F(b_1^\dagger,\dots,b_N^\dagger)|0\rangle_a\otimes|0\rangle_b\ , \tag{3.2.20}$$

where $F(b_1^\dagger,\dots,b_N^\dagger)$ is a homogeneous polynomial of N variables, and which is either symmetric (for bosons) or antisymmetric (for fermions) under exchange of any two variables. We use Deg$[F]$ to denote the total degree of the polynomial function F. Then if we scale all arguments of F by a numerical factor λ, we have

$$F(\lambda b_1^\dagger,\dots,\lambda b_N^\dagger) = \lambda^{\text{Deg}[F]}F(b_1^\dagger,\dots,b_N^\dagger)\ . \tag{3.2.21}$$

Let $\mathcal{N}_b = \sum_{j=1}^N b_j^\dagger b_j$ be the total number operator for the N guiding center ladder operators. Then the homogeneity property of F implies that $|\psi_0\rangle$ is an eigenvalue of $\mathcal{N}_b$ with eigenvalue Deg$[F]$.

To compute η_H^{ab} for these trial FQH states we use a connection between the APD generators and the generators of the group $SU(1,1)$ (see, for example, Ref. [124]). Define the operators

$$K_0 = \frac{1}{2}\sum_{j=1}^{N}\left(b_j^\dagger b_j + \frac{1}{2}\right) \tag{3.2.22a}$$

$$K_+ = \frac{1}{2}\sum_{j=1}^{N}(b_j^\dagger)^2 \tag{3.2.22b}$$

$$K_- = \frac{1}{2}\sum_{j=1}^{N}(b_j)^2\ . \tag{3.2.22c}$$

These operators obey the commutation relations of the Lie algebra of the group $SU(1,1)$,

$$
\begin{aligned}
[K_0, K_\pm] &= \pm K_\pm && \text{(3.2.23a)} \\
[K_-, K_+] &= 2K_0 \; . && \text{(3.2.23b)}
\end{aligned}
$$

The Fock space of the oscillators b_j forms a (reducible) representation of this algebra, and the generators Λ^{ab} can be expressed in terms of the $SU(1,1)$ generators as

$$
\begin{aligned}
\Lambda^{11} &= K_0 + \frac{1}{2}K_+ + \frac{1}{2}K_- && \text{(3.2.24)} \\
\Lambda^{22} &= K_0 - \frac{1}{2}K_+ - \frac{1}{2}K_- && \text{(3.2.25)}
\end{aligned}
$$

and

$$
\Lambda^{12} = \Lambda^{21} = \frac{-i}{2}(K_- - K_+) \; . \tag{3.2.26}
$$

It is clear that the state $|\psi_0\rangle$ is an eigenstate of K_0 with eigenvalue $\frac{1}{2}(\text{Deg}[F] + \frac{N}{2})$. It then follows that the expectation values $\langle\psi_0|K_\pm|\psi_0\rangle$ are equal to zero as $K_\pm|\psi_0\rangle$ is orthogonal to $|\psi_0\rangle$. Then, for the trial state parametrized by the function F, we have

$$
\langle\psi_0|\Lambda^{ab}|\psi_0\rangle = \frac{1}{2}\left[\text{Deg}[F] + \frac{N}{2}\right]\delta^{ab} \; . \tag{3.2.27}
$$

A similar computation shows that for a trial state in the n^{th} Landau level we have

$$
\langle\psi_0|\tilde{\Lambda}^{ab}|\psi_0\rangle = \frac{1}{2}\left(nN + \frac{N}{2}\right)\delta^{ab} \; , \tag{3.2.28}
$$

which follows since the product $\prod_{j=1}^{N} \frac{(a_j^\dagger)^n}{\sqrt{n!}}$ is a homogeneous polynomial in the $a_j^\dagger$ of total degree nN.

For the case of the $\nu = \frac{1}{m}$ Laughlin state (m a positive integer) we have

$$
F(b_1^\dagger, \ldots, b_N^\dagger) = \prod_{j<k}(b_j^\dagger - b_k^\dagger)^m \; , \tag{3.2.29}
$$

and so

$$
\text{Deg}[F] = \frac{1}{2}mN(N-1) \; . \tag{3.2.30}
$$

If we consider this Laughlin state in the lowest Landau level ($n = 0$) then we find that

$$\langle\psi_0|\Lambda^{ab}|\psi_0\rangle = \frac{1}{2}\left[\frac{1}{2}mN^2 + \left(\frac{1-m}{2}\right)N\right]\delta^{ab} \tag{3.2.31a}$$

$$\langle\psi_0|\tilde{\Lambda}^{ab}|\psi_0\rangle = \frac{N}{4}\delta^{ab} \; , \tag{3.2.31b}$$

and so

$$\eta_H^{ab} = -\frac{\hbar}{A}\frac{1}{2}\left[\frac{1}{2}mN^2 + \left(\frac{1-m}{2}\right)N\right]\delta^{ab} \; , \tag{3.2.32}$$

while

$$\tilde{\eta}_H^{ab} = \frac{\hbar}{4}\frac{N}{A}\delta^{ab} \; . \tag{3.2.33}$$

Both of these tensors are proportional to the identity matrix (in this rotation-invariant case), and it is convenient to denote the constants of proportionality by

$$\eta_H = -\frac{\hbar}{A}\frac{1}{2}\left[\frac{1}{2}mN^2 + \left(\frac{1-m}{2}\right)N\right] \tag{3.2.34}$$

and

$$\tilde{\eta}_H = \frac{\hbar}{4}\frac{N}{A} \tag{3.2.35}$$

so that we can simply write $\eta_H^{ab} = \eta_H\delta^{ab}$ and similarly for $\tilde{\eta}_H^{ab}$.

For a Laughlin FQH droplet with $\nu = \frac{1}{m}$, and consisting of a large number N of particles, we have $A \approx 2\pi\ell_B^2 mN$. Then, in its current form, the coefficient η_H in the guiding center Hall viscosity tensor is the sum of an extensive (order N) term and an intensive (order 1) term. Since A itself is proportional to N, the extensive term in η_H comes from the superextensive (order N^2) term in $\langle\psi_0|\Lambda^{ab}|\psi_0\rangle$. This term is associated with a uniform rotational motion (in fact, it is just the orbital angular momentum) of the FQH fluid, and so it has been argued that one should subtract this term when defining the guiding center Hall viscosity [100, 101]. If we make this subtraction then we end up with the regularized quantities

$$\langle\psi_0|\Lambda^{ab}|\psi_0\rangle_{reg} \;=\; \frac{1}{2}\left[\left(\frac{1-m}{2}\right)N\right]\delta^{ab} \tag{3.2.36}$$

$$\eta_{H,reg} \;=\; -\frac{\hbar}{2}\left(\frac{1-m}{2}\right)\rho_0 \; , \tag{3.2.37}$$

where $\rho_0 = \frac{1}{2\pi\ell_B^2 m} = \frac{N}{A}$ is the density of the $\nu = \frac{1}{m}$ Laughlin FQH state at large N. We discuss the physical interpretation of this regularization scheme in the context of the CSMM in Sec. 3.7.

The Landau orbit contribution $\tilde{\eta}_H$ does not require regularization as it only consists of an intensive term. In terms of the density ρ_0 of the Laughlin state this coefficient has the form

$$\tilde{\eta}_H = \frac{\hbar\rho_0}{4} \ . \tag{3.2.38}$$

Then the full Hall viscosity coefficient for the $\nu = \frac{1}{m}$ Laughlin state (in the lowest Landau level and after regularization of the guiding center part) is

$$\eta_{tot} = \tilde{\eta}_H + \eta_{H,reg} = \frac{\hbar\rho_0 m}{4} \ , \tag{3.2.39}$$

as originally found by Read [87]. It is interesting to observe that since $\rho_0 = \frac{1}{2\pi\ell_B^2 m}$, the full Hall viscosity coefficient η_{tot} actually does not depend on the filling fraction of the Laughlin state (i.e., it does not depend on m).

The coefficient $\frac{1-m}{2}$ appearing in $\eta_{H,reg}$ is what Haldane has termed the "guiding center spin" of a FQH state. This coefficient has been denoted as "$\overline{s}$" in Ref. [100] and "s" in Ref. [101]. It is also equal to minus the "anisospin" defined in Refs. [112, 113], and denoted there by ς. We choose to adopt the notation of Refs. [112, 113] and so we write

$$\eta_{H,reg} = \frac{\hbar}{2}\varsigma\rho_0 \tag{3.2.40}$$

with $\varsigma = \frac{m-1}{2}$. We see that unlike the full Hall viscosity coefficient η_{tot}, the guiding center contribution to the Hall viscosity has a clear dependence on m. It follows that different Laughlin states cannot be distinguished by their full Hall viscosity η_{tot}, but they *can* be distinguished by their guiding center Hall viscosity $\eta_{H,reg}$ which, moreover, has been argued to be connected to the physical property of intrinsic electric dipole moment at the edge of the FQH state [101].

3.3 Noncommutative Chern-Simons theory

In this section we review Susskind's noncommutative Chern-Simons (NCCS) theory description of the Laughlin FQH states [27]. This will pave the way for the discussion of the Chern-Simons matrix model in the next section, as the Chern-Simons matrix model can be thought of as a particular regularization of the NCCS theory. To prepare the reader for this discussion in this section we first make a few remarks about the two different formulations ("operator" vs. "star product" formulations) of noncommutative field theory. We then present the NCCS theory in both formulations. Finally, we discuss the NCCS theory in the limit of weak noncommutativity, and its connection with the dynamics of a fluid of charged particles in a magnetic field. From this connection one sees that the full NCCS theory should be understood as describing a fluid of charged particles in a magnetic field on a noncommutative space. Our discussion of noncommutative field theory closely follows that in Refs. [126–128]. For the fluid picture of the NCCS theory we follow Refs. [27, 129]. Readers who are already familiar with noncommutative field theory and the NCCS theory may want to skip this section.

3.3.1 Two formulations of noncommutative field theory

Consider a classical field theory in $2+1$ dimensions in which the two-dimensional space is taken to be $\mathbb{R}^2$, and let $\mathbf{x}=(x^1,x^2)$ denote the spatial coordinates. We denote a general field in this theory as $\Phi(t,\mathbf{x})$. In such a field theory the fields $\Phi(t,\mathbf{x})$ at a fixed time t are elements of the ordinary algebra of functions on $\mathbb{R}^2$ (the commutative algebra generated by pointwise addition and multiplication of functions of $\mathbf{x}$). The noncommutative deformation of the this theory that we consider consists of replacing the ordinary space $\mathbb{R}^2$ with a "noncommutative plane" whose two spatial coordinates do not commute with each other. The time direction will always be commutative in this Chapter, i.e., we consider theories in two noncommutative spatial dimensions and one commutative (or ordinary) time direction.

In the noncommutative deformation of the classical field theory, the fields (again at a fixed time t) instead take values in the algebra $\mathbb{R}^2_\theta$ which consists of all complex linear combinations of products of position variables $\hat{x}^a$, $a=1,2$, satisfying the commutation relation

$$[\hat{x}^1,\hat{x}^2]=i\theta\ . \tag{3.3.1}$$

Here θ is a constant real number with dimensions of length squared; it controls the "strength" of the noncommutativity of this theory. The algebra $\mathbb{R}^2_\theta$ comes equipped with a conjugation operator "$\dagger$" (which one can think of as Hermitian conjugation), and the operators $\hat{x}^a$ are assumed to be invariant under this operation[5]. We see that the algebra $\mathbb{R}^2_\theta$ is nothing but the universal enveloping algebra of the Heisenberg algebra specified by $\hat{x}^a$ and the commutation relation of Eq. (3.3.1). The operators $\hat{x}^a$ are sometimes said to be coordinates on a "noncommutative plane". In the noncommutative theory the notion of a point no longer makes sense, and the smallest area that one can resolve is of order θ.

In the noncommutative field theory, the notion of integration over space is replaced with a trace in a representation of the Heisenberg algebra of the noncommutative coordinates $\hat{x}^a$. Usually this representation is taken to be the Fock representation in which the ladder operators

$$\hat{a} = \frac{1}{\sqrt{2\theta}}(\hat{x}^1+i\hat{x}^2) \tag{3.3.2}$$

$$\hat{a}^\dagger = \frac{1}{\sqrt{2\theta}}(\hat{x}^1-i\hat{x}^2) \tag{3.3.3}$$

act on a Fock space $\mathcal{H}_F$ generated by the action of the raising operator $\hat{a}^\dagger$ on a vacuum state $|0\rangle$ which is annihilated

[5] For any complex number c and any $y\in\mathbb{R}^2_\theta$ we have $(cy)^\dagger=\overline{c}y^\dagger$, where $\overline{c}$ is the complex conjugate of c.

by the lowering operator $\hat{a}$. The action functional for the noncommutative field theory then takes the form

$$S = \int dt \, \mathrm{Tr}_{\mathcal{H}_F} \left\{ (\cdots) \right\} \tag{3.3.4}$$

where $(\cdots)$ denotes a Lagrangian written in terms of fields $\hat{\Phi}(t)$ which are operators on the space $\mathcal{H}_F$, and whose matrix elements are functions of time.

It is natural to call the formulation of noncommutative field theory that we have just described the "operator formulation." We now describe an alternative formulation, which one might call the "star-product formulation," which may be more familiar to some readers. In this formulation one instead works with fields $\Phi(t, \mathbf{x})$ which are ordinary functions of the coordinates $\mathbf{x}$ on $\mathbb{R}^2$, but replaces the ordinary product of functions with the *Groenewold-Moyal star product* "$\star$", which is defined as follows. For any two functions $f(\mathbf{x})$ and $g(\mathbf{x})$ of $\mathbf{x}$ we have

$$\begin{aligned} f(\mathbf{x}) \star g(\mathbf{x}) &= e^{\frac{i}{2}\theta\epsilon^{ab}\frac{\partial}{\partial y^a}\frac{\partial}{\partial z^a}} f(\mathbf{y}) g(\mathbf{z}) \Big|_{\mathbf{y}=\mathbf{z}=\mathbf{x}} \\ &= f(\mathbf{x}) g(\mathbf{x}) + \frac{i}{2}\theta\epsilon^{ab}\partial_a f(\mathbf{x}) \partial_b g(\mathbf{x}) + \dots \,, \end{aligned} \tag{3.3.5}$$

and where in the last line the ellipses denote terms of order θ^2 and higher. For two functions $f(\mathbf{x})$ and $g(\mathbf{x})$ which vanish at spatial infinity we have the important property that

$$\int d^2\mathbf{x} \, f(\mathbf{x}) \star g(\mathbf{x}) = \int d^2\mathbf{x} \, f(\mathbf{x}) g(\mathbf{x}) \,, \tag{3.3.6}$$

which follows after integration by parts on the higher derivative terms in the star product. There is no analogous result for integrals of star products of three or more functions.

These two formulations of noncommutative field theory are related by the *Wigner-Weyl* mapping of functions and operators. This mapping is as follows. Let $f(\mathbf{x})$ be an ordinary function on $\mathbb{R}^2$ and let

$$\tilde{f}(\mathbf{k}) = \int d^2\mathbf{x} \, f(\mathbf{x}) e^{-ik_a x^a} \tag{3.3.7}$$

be its Fourier transform. Then we can define a Weyl-ordered operator $\hat{f}$ by taking the inverse Fourier transform but replacing x^a with $\hat{x}^a$ in the exponential,

$$\hat{f} = \int \frac{d^2\mathbf{k}}{(2\pi)^2} \, \tilde{f}(\mathbf{k}) e^{ik_a \hat{x}^a} \,. \tag{3.3.8}$$

One can check that this mapping satisfies the following properties which will be needed later:

$$\hat{f}\hat{g} = \widehat{f \star g} \tag{3.3.9}$$

$$\mathrm{Tr}_{\mathcal{H}_F}\left\{\hat{f}\right\} = \frac{1}{2\pi\theta}\int d^2\mathbf{x}\, f(\mathbf{x})\,. \tag{3.3.10}$$

To check the second property one can express the trace over $\mathcal{H}_F$ using a basis $\{|x^1\rangle\}$ of eigenstates of $\hat{x}^1$ as

$$\mathrm{Tr}_{\mathcal{H}_F}\left\{\hat{f}\right\} = \int_{-\infty}^{\infty} dx^1\, \langle x^1|\hat{f}|x^1\rangle \tag{3.3.11}$$

and then plug in the expression Eq. (3.3.8) for $\hat{f}$.

The Chern-Simons matrix model that we study below is a particular regularization of the NCCS theory in its operator formulation. Therefore, for our purposes we generally find that the operator formulation of the NCCS theory is more convenient. However, the star product formulation is still useful for the study of the behavior of the theory near the commutative limit $\theta \to 0$, and so we will have occasion to use both formulations of the NCCS theory in what follows.

3.3.2 NCCS theory in the operator formulation

We now review the operator formulation of the NCCS theory. In the operator formulation, the NCCS theory consists of three fields $\hat{X}^a(t)$, $a = 1, 2$, and $\hat{A}_0(t)$. All fields should be thought of as operators on the Fock space $\mathcal{H}_F$ whose matrix elements are functions of time. In addition, all fields are Hermitian (i.e., all fields are invariant under the "†" operation on the algebra $\mathbb{R}^2_\theta$). We also consider the theory on a time interval of length T and assume periodic boundary conditions in time so that $\hat{X}^a(0) = \hat{X}^a(T)$, and likewise for $\hat{A}_0(t)$. In addition to the noncommutativity parameter θ, the theory includes various coupling constants including $e > 0$, an electric charge, and $B > 0$, a constant magnetic field. We discuss the physical interpretation of this theory as representing a charged fluid in a magnetic field later in this section (and we will see that the charge of the particles which make up this fluid is actually $q = -e < 0$).

The action for the NCCS theory in the operator formulation takes the form

$$S_{NCCS} = -\frac{eB}{2}\int_0^T dt\, \mathrm{Tr}_{\mathcal{H}_F}\left\{\epsilon_{ab}\hat{X}^a D_0 \hat{X}^b + 2\theta\hat{A}_0\right\}\,, \tag{3.3.12}$$

where we introduced a covariant derivative

$$D_0\hat{X}^b = \dot{\hat{X}}^b + i[\hat{X}^b, \hat{A}_0]\,. \tag{3.3.13}$$

and where the dot denotes a time derivative. The field $\hat{A}_0$ functions as a Lagrange multiplier and its equation of motion yields the constraint

$$[\hat{X}^1, \hat{X}^2] = i\theta \; . \tag{3.3.14}$$

This constraint can only be satisfied by operators $\hat{X}^a$ on an infinite-dimensional space. This is due to the fact that if the variables $\hat{X}^a$ were finite-dimensional matrices, then the trace of the left-hand side of the equation is zero while the trace of the right-hand side would be proportional to the size of the matrices. The CSMM discussed in the next section is a modification of the NCCS theory which features a modified constraint that can be satisfied by operators (matrices) on a finite-dimensional space.

If we ignore the term containing $2\theta\hat{A}_0$ for a moment, then one can check that the action is invariant under the gauge transformation

$$\hat{X}^a \;\; \rightarrow \;\; \hat{V}\hat{X}^a\hat{V}^\dagger \tag{3.3.15a}$$

$$\hat{A}_0 \;\; \rightarrow \;\; \hat{V}\hat{A}_0\hat{V}^\dagger + i\hat{V}\dot{\hat{V}}^\dagger, \tag{3.3.15b}$$

where $\hat{V}(t)$ is an arbitrary time-dependent unitary operator on the Fock space $\mathcal{H}_F$. In particular, this follows from the fact that, under this transformation, the covariant derivative transforms as $D_0\hat{X}^b \rightarrow \hat{V}D_0\hat{X}^b\hat{V}^\dagger$. To understand these gauge transformations in the presence of the term $2\theta\hat{A}_0$, we need to constrain the allowed $\hat{V}$'s that we consider [114]. To motivate this restriction we now briefly discuss some aspects of the geometry of the noncommutative plane.

Consider the occupation number basis $\{|n\rangle\}_{n\in\mathbb{N}}$ of the Fock space $\mathcal{H}_F$ ($|n\rangle \propto (\hat{a}^\dagger)^n|0\rangle$). The radius squared operator $\hat{R}^2 = \delta_{ab}\hat{x}^a\hat{x}^b$ is diagonal in this basis and we have $\hat{R}^2|n\rangle = 2\theta(n+\frac{1}{2})|n\rangle$. Thus, the occupation number n can be identified with the distance squared from the origin in the noncommutative plane. We now restrict our attention to gauge transformations defined by unitary operators $\hat{V}(t)$ which act as the identity on states $|n\rangle$ with n sufficiently large, say $n > N_0$. The actual value of N_0 is not important for the argument. This is the noncommutative analogue of requiring gauge transformations in a commutative gauge theory on the space $\mathbb{R}^2$ to tend to the identity at spatial infinity.

With this restriction on possible gauge transformations, the unitary operator $\hat{V}(t)$ defines a map from the periodic time interval $[0, T)$ to $U(N_0)$, the group of unitary matrices of size N_0. Large gauge transformations are those $\hat{V}(t)$ which correspond to a nontrivial element of the homotopy group $\pi_1(U(N_0)) = \mathbb{Z}$. The full NCCS action is not invariant under these large gauge transformations because of the presence of the $2\theta\hat{A}_0$ term. In Ref. [114], Polychronakos and Nair have shown that requiring the exponential $e^{i\frac{S_{CSMM}}{\hbar}}$ to be invariant under these large gauge transformations enforces a quantization rule on θ which states that

$$eB\theta = \hbar m,\ m \in \mathbb{Z}\ , \tag{3.3.16}$$

or

$$\theta = \ell_B^2 m\ ,\ m \in \mathbb{Z}\ , \tag{3.3.17}$$

where $\ell_B^2 = \frac{\hbar}{eB}$ is the square of the magnetic length defined earlier. This quantization rule is the noncommutative analogue of the level quantization which obtains in ordinary (say $SU(N)$) Chern-Simons theory on a commutative space.

3.3.3 NCCS theory in the star product formulation

We now discuss the NCCS theory in the star product formulation. In this form the theory looks very similar to the ordinary Chern-Simons theory (i.e., Chern-Simons theory on the commutative space $\mathbb{R}^2$). We proceed by deriving the star product formulation of the NCCS theory from the operator formulation by using the Wigner-Weyl mapping discussed earlier in this section. To do this we need to know how spatial derivatives are represented in the operator formulation of the theory. Derivative operators $\hat{\partial}_a$ in the operator formulation of noncommutative field theory are defined by

$$\hat{\partial}_1 = \frac{i\hat{x}^2}{\theta}\ ,\ \hat{\partial}_2 = -\frac{i\hat{x}^1}{\theta} \tag{3.3.18}$$

and one can check that

$$[\hat{\partial}_a, \hat{x}^b] = \delta_a^b\ , \tag{3.3.19}$$

just as one has for ordinary derivatives of functions on $\mathbb{R}^2$. In addition, in the Wigner-Weyl mapping one has

$$[\hat{\partial}_a, \hat{f}] = \widehat{\partial_a f}\ , \tag{3.3.20}$$

so under this mapping the ordinary derivative of a function $f(\mathbf{x})$ with respect to x^a is mapped to the commutator of $\hat{\partial}_a$ with $\hat{f}$ (i.e., the *adjoint* action of $\hat{\partial}_a$ on $\hat{f}$).

The first step towards deriving the star product formulation of NCCS theory is to make a change of variables in the operator formulation by defining two new fields $\hat{A}_a$, $a = 1, 2$, which are related to the fields $\hat{X}^a$ by

$$\hat{X}^a = \hat{x}^a + \theta\epsilon^{ab}\hat{A}_b\ . \tag{3.3.21}$$

Under a gauge transformation the new fields transform as[6]

$$\hat{A}_a \to \hat{V}\hat{A}_a\hat{V}^\dagger + i\hat{V}[\hat{\partial}_a, \hat{V}^\dagger] \,. \tag{3.3.22}$$

This transformation resembles the transformation of an ordinary non-Abelian gauge field. In addition, in the new variables, the NCCS constraint of Eq. (3.3.14) becomes

$$\hat{F}_{12} = 0 \,, \tag{3.3.23}$$

where we defined the field strength for noncommutative gauge theory as

$$\hat{F}_{ab} = [\hat{\partial}_a, \hat{A}_b] - [\hat{\partial}_b, \hat{A}_a] - i[\hat{A}_a, \hat{A}_b] \,. \tag{3.3.24}$$

Thus, the constraint in NCCS theory is an exact noncommutative analogue of the constraint enforced by the temporal component of the gauge field in ordinary Chern-Simons theory on a commutative space.

After tedious algebra (including many uses of the cyclic property of the trace) one can show that after performing this transformation the NCCS action takes the form

$$S_{NCCS} = -\frac{eB\theta^2}{2}\int_0^T dt\, \mathrm{Tr}_{\mathcal{H}_F}\left\{\epsilon^{ab}\hat{A}_a\dot{\hat{A}}_b - \epsilon^{ab}\hat{A}_0[\hat{\partial}_a, \hat{A}_b] + \epsilon^{ab}\hat{A}_b[\hat{\partial}_a, \hat{A}_0] + \frac{2i}{3}\epsilon^{\mu\nu\lambda}\hat{A}_\mu\hat{A}_\nu\hat{A}_\lambda\right\} , \tag{3.3.25}$$

where the Greek indices μ, ν, λ run over the range $0, 1, 2$. There is one subtle point in the derivation of this equation which involves a term which is a total time derivative. Specifically, after the transformation from the $\hat{X}^a$ variables to the $\hat{A}_a$ variables one finds a term

$$-\frac{eB}{2}\int_0^T dt\, \mathrm{Tr}_{\mathcal{H}_F}\left\{-\theta\hat{x}^a\dot{\hat{A}}_a\right\} \,. \tag{3.3.26}$$

Since $\hat{x}^a$ has no time dependence this term is a total derivative, and it evaluates to zero since we assumed periodic boundary conditions on all fields in the time direction.

Finally, we apply the Wigner-Weyl mapping to write the NCCS action in the star product formulation as

$$S_{NCCS} = \frac{eB\theta}{4\pi}\int_0^T dt \int d^2\mathbf{x}\, \epsilon^{\mu\nu\lambda}\Big(A_\mu \star \partial_\nu A_\lambda - \frac{2i}{3}A_\mu \star A_\nu \star A_\lambda\Big) \,. \tag{3.3.27}$$

The quantization condition on θ (Eq. (3.3.17)) then implies that the coefficient of the action is

$$\frac{eB\theta}{4\pi} = \frac{\hbar m}{4\pi} \,. \tag{3.3.28}$$

[6]This is derived by requiring the gauge transformation of $\hat{x}^a + \theta\epsilon^{ab}\hat{A}_b$ to coincide with the gauge transformation of $\hat{X}^a$ from Eq. (3.3.15).

Then, in units where $\hbar = 1$, we find the NCCS action at level $m \in \mathbb{Z}$. If we take $\ell_B^2 \to 0$, which also sends $\theta \to 0$, then we recover the ordinary $U(1)$ Chern-Simons theory at level m (again with $\hbar = 1$ for now),

$$S_{CS} = \frac{m}{4\pi} \int_0^T dt \int d^2\mathbf{x}\, \epsilon^{\mu\nu\lambda} A_\mu \partial_\nu A_\lambda \,. \tag{3.3.29}$$

For completeness we note here that in the star product formulation the noncommutative field strength is

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - i(A_\mu \star A_\nu - A_\nu \star A_\mu) \,, \tag{3.3.30}$$

and the equation of motion of the NCCS theory is equivalent to $F_{\mu\nu} = 0$, just like in ordinary Chern-Simons theory.

3.3.4 Fluid interpretation of the NCCS theory at small θ

We now discuss the behavior of the NCCS theory in the limit of weak noncommutativity in which θ is assumed to be small. Note that since θ has units, and since there is no other length scale in the problem to compare θ to, it is more accurate to say that in this section we study a *truncation* of the NCCS theory at first order in θ. In the star product formulation of the theory this truncation simply amounts to neglecting terms of order θ^2 and higher in the star product of functions. In this limit we will see that the NCCS theory has an interpretation as describing a fluid of charged particles in a constant magnetic field B, as was discussed by Susskind [27] (see also Refs. [129, 130]).

To consider the NCCS theory in the regime of small θ we start by using the cyclic property of the trace to write the action in the form

$$S_{NCCS} = -\frac{eB}{2} \int_0^T dt\, \mathrm{Tr}_{\mathcal{H}_F} \left\{ \epsilon_{ab} \hat{X}^a \dot{\hat{X}}^b + 2\hat{A}_0 \left(\theta + i[\hat{X}^1, \hat{X}^2] \right) \right\} . \tag{3.3.31}$$

We then use the Wigner-Weyl mapping, and keep only the terms up to order θ in the star product, to find that in the limit of small θ

$$S_{NCCS} \to -\frac{eB}{2} \frac{1}{2\pi\theta} \int_0^T dt \int d^2\mathbf{x} \left(\epsilon_{ab} X^a \dot{X}^b + 2\theta A_0 (1 - \{X^1, X^2\}) \right) , \tag{3.3.32}$$

where for any $f(\mathbf{x})$ and $g(\mathbf{x})$ we have the Poisson bracket

$$\{f, g\} = \epsilon^{ab} \partial_a f \partial_b g \,. \tag{3.3.33}$$

Susskind observed that in this limit the NCCS theory describes the dynamics of a charged fluid at constant density $\rho_0 = \frac{1}{2\pi\theta}$ in a constant magnetic field B, and in the limit where the cyclotron frequency is sent to infinity. In fact, in Susskind's original derivation he starts with the fluid description and then observes that it coincides with the small θ

limit of the NCCS theory. We now briefly remind the reader of this connection between the NCCS theory and fluid dynamics.

The starting point is the *Lagrange* description[7] of a fluid of charged particles moving on the plane $\mathbb{R}^2$ in a background electromagnetic field. In the Lagrange description of a fluid one keeps track of the motion of the individual particles in the fluid, and measures their current position with respect to some reference configuration. In this description we use coordinates $\mathbf{x}$ to describe the reference configuration of the fluid and coordinates $X^a(t,\mathbf{x})$, $a = 1,2$, to describe the configuration of the fluid at a later time t. Without loss of generality, we may assume that $X^a(0,\mathbf{x}) = x^a$. Thus, $X^a(t,\mathbf{x})$ is the position, at time t, of the fluid particle which was at position x^a at $t = 0$. We also assign a constant density ρ_0 to the fluid in the reference configuration.

The action for a Lagrange fluid made up of particles of mass M and charge q in the presence of a background electromagnetic field takes the form

$$S = \int_0^T dt \int d^2\mathbf{x}\, \rho_0 \left(\frac{1}{2} M \delta_{ab} \dot{X}^a \dot{X}^b + q\mathcal{A}_a(t,\mathbf{X})\dot{X}^a - q\varphi(t,\mathbf{X}) \right) , \tag{3.3.34}$$

where $\mathcal{A}_a(t,\mathbf{X})$ and $\varphi(t,\mathbf{X})$ are the vector and scalar potentials, respectively, for the external electromagnetic field. Intuitively, this action is just the sum over all particles in the fluid of the ordinary action for a massive charged particle in a background electromagnetic field. However, the discrete sum over particle labels has been replaced with an integration over the reference coordinates $\mathbf{x}$ weighted with the density ρ_0 in the reference configuration. The reference coordinates $\mathbf{x}$ can therefore be considered as a set of continuous particle labels.

To see the connection of the fluid model to the NCCS theory we first place the system in a uniform background magnetic field with strength $B > 0$. This can be accomplished by setting $\varphi(t,\mathbf{X}) = 0$ and

$$\mathcal{A}_a(t,\mathbf{X}) = -\frac{B}{2}\epsilon_{ab}X^b , \tag{3.3.35}$$

where we have chosen the symmetric gauge for the vector potential. Next, we set the mass of the particles to zero, $M = 0$. This corresponds to taking the cyclotron frequency $\omega_c = \frac{eB}{M}$ to infinity, which is similar to a projection into the lowest Landau level (since $\hbar\omega_c$ is the energy gap between Landau levels). Finally, we take the charge of the particles to be $q = -e$ with $e > 0$. Then at this point the action reads as

$$S = -\frac{eB}{2}\rho_0 \int_0^T dt \int d^2\mathbf{x}\, \epsilon_{ab}X^a\dot{X}^b . \tag{3.3.36}$$

Note that ρ_0 can be pulled out of the integral since we assumed it was constant. We also mention here that our

[7] The relation between noncommutative gauge theory and the Lagrange description of a fluid is discussed in detail in Ref. [130].

conventions for the direction of the magnetic field and the charge of the particles in the fluid exactly matches our conventions for the setup of the quantum Hall problem from Sec. 3.2.

The next step is to incorporate a Lagrange multiplier which enforces the constraint that the fluid remains at the constant density ρ_0 at all times. The density $\rho(t, \mathbf{X})$ of the fluid at time t is related to the initial density ρ_0 by the Jacobian $\epsilon^{ab}\frac{\partial X^1}{\partial x^a}\frac{\partial X^2}{\partial x^b}$ of the map from the reference coordinates to the fluid coordinates $\mathbf{X}$ at time t as

$$\rho(t, \mathbf{X})\epsilon^{ab}\frac{\partial X^1}{\partial x^a}\frac{\partial X^2}{\partial x^b} = \rho_0 \ . \tag{3.3.37}$$

Then the constraint that $\rho(t, \mathbf{X}) = \rho_0$ for all t can be written in terms of a Poisson bracket as

$$\{X^1, X^2\} = 1 \ . \tag{3.3.38}$$

We denote the Lagrange multiplier enforcing this constraint by $A_0(t, \mathbf{x})$, and write the action with the constraint included in the form

$$S = -\frac{eB}{2}\rho_0 \int_0^T dt \int d^2\mathbf{x} \left(\epsilon_{ab} X^a \dot{X}^b + 2\theta A_0 \left(1 - \{X^1, X^2\}\right) \right) , \tag{3.3.39}$$

where we have introduced a parameter θ with units of (length)^2. With this choice, the Lagrange multiplier field A_0 has units of (time)$^{-1}$.

We can now see that the small θ limit of the NCCS action from Eq. (3.3.32) is exactly the action for a fluid of particles with charge $q = -e$ at the constant density $\rho_0 = \frac{1}{2\pi\theta}$ in a constant background magnetic field B in the limit in which the cyclotron frequency is taken to infinity. This limit is analogous to the projection into the lowest Landau level, and it is the physical reason why this fluid theory (and the NCCS theory) is expected to describe FQH physics in the lowest Landau level [27]. In the full NCCS theory we should then interpret the fields $\hat{X}^a(t)$ as describing the positions of particles in a fluid on a noncommutative space, as discussed by Susskind [27] (see also Ref. [129] for a review of the physics of such *noncommutative fluids*).

3.4 The Chern-Simons Matrix Model

In this section we discuss the Chern-Simons matrix model (CSMM), which was introduced by Polychronakos in Ref. [29]. This model can be thought of as a particular regularization of the operator formulation of the NCCS theory, in which the fields $\hat{X}^a(t)$ (which were operators on the *infinite-dimensional* Fock space $\mathcal{H}_F$) are now finite $N \times N$ matrices $X^a(t)$ instead. Note that we *do not* use a hatted notation for the finite size matrix variables of the NCCS theory. The parameter N serves as a regulator which should be taken to infinity to recover the NCCS theory discussed in the previous section. The fluid interpretation of the NCCS theory carries over to the CSMM, so we still interpret the matrix variables $X^a(t)$ as representing the coordinates of particles in a fluid on a noncommutative space, only now the fluid turns out to occupy a finite area of this space. In other words, the CSMM is a model of a finite droplet *droplet* of noncommutative fluid.

Since the CSMM can be difficult to understand, we begin this section by making a few remarks about our notation and conventions, and then discuss some subtleties of this model. We then review the quantization of this model following Refs. [29, 117]. Finally, we review (following the original discussion in Ref. [29]) the calculation of the area A and density ρ_0 of the droplet of noncommutative fluid represented by the ground state of the CSMM. We will then be able to identify the CSMM having $\theta = \ell_B^2 m$ as describing the $\nu = \frac{1}{m}$ Laughlin state by comparing the results for ρ_0 and A to the known answers for a droplet of FQH fluid in the $\nu = \frac{1}{m}$ Laughlin state in the limit of a large number of particles N.

3.4.1 Some remarks on notation

The CSMM, and especially the quantization of this model, can be quite tricky due to two separate noncommutative structures which appear. First, at the classical level the degrees of freedom in this model are Hermitian $N \times N$ matrix variables X^1, X^2, A_0, as well as a complex vector Ψ of length N. All of these variables are functions of time. Since some of the variables are matrix variables, ordinary (i.e., classical) matrix multiplication of these variables is not commutative. Next, upon quantization of the model, the *matrix elements* of X^1, X^2, and A_0 (and also the components of Ψ) become *operators* on a separate Hilbert space, which is unrelated to the vector space on which the classical matrix variables act. Thus, in the quantized matrix model there are two sources of noncommutativity. The first source is the fact that we are dealing with matrix variables from the start, and the second source comes from the fact that the matrix elements of the original matrix variables are now operators on a second Hilbert space, and so multiplication of individual matrix elements does not commute either, but for a different reason.

In an attempt to present this model in as clear a manner as possible, we will adhere to the following notational conventions. First, we use $[\cdot,\cdot]_M$ to denote a matrix commutator of classical matrices, and use $[\cdot,\cdot]$ (with no subscript) to denote the commutator of quantum operators. We also reserve the symbol $\dagger$ to denote Hermitian conjugation of quantum operators. In all manipulations with classical matrix variables, we instead use an overline to denote complex conjugation of a matrix and a superscript 'T' to denote a transpose. So if A is an $N \times N$ matrix variable, then $\overline{A}^T$ is its transpose conjugate, i.e., if A has matrix elements A_{jk}, then the matrix elements of $\overline{A}^T$ are $(\overline{A}^T)_{jk} = \overline{A}_{kj}$ (and Hermitian matrices satisfy the relation $\overline{A}^T = A$). As we mentioned before, in the quantum theory the matrix elements A_{jk} are promoted to operators on a Hilbert space. We denote the Hermitian conjugate (with respect to the inner product on this Hilbert space) of the operator A_{jk} by $A^{\dagger}_{jk}$. Note that for a generic matrix variable A it is entirely possible that the operator $A^{\dagger}_{jk}$ is not the same as the operator $(\overline{A}^T)_{jk}$. In what follows we also make every effort to avoid using 'i' as an index, and instead try to reserve it for the symbol meaning $\sqrt{-1}$, and occasionally for the differential geometry operation $i_{\underline{v}}$ of interior multiplication by a vector field $\underline{v}$.

3.4.2 Description of the model

In this subsection we describe the CSMM of the Laughlin quantum Hall states [29]. The degrees of freedom in this model are two $N \times N$ matrices $X^a(t)$, $a = 1, 2$, an $N \times N$ matrix $A_0(t)$, and a complex vector $\Psi(t)$ of length N. All degrees of freedom depend on time. The matrices $X^a(t)$ and $A_0(t)$ are all Hermitian and so they have real eigenvalues. The variables X^a are to be interpreted as coordinates in the Lagrange description of a fluid on the noncommutative plane, in accordance with the physical ideas of Susskind and Polychronakos [27, 29](and as we reviewed at the end of Sec. 3.3). The number N will later be identified with the number of electrons in a Landau level. The action for the CSMM takes the form

$$S_{CSMM} = -\frac{eB}{2}\int_0^T dt\ \mathrm{Tr}\Big\{\epsilon_{ab}X^a D_0 X^b + 2\theta A_0 + \tilde{\omega}\delta_{ab}X^a X^b\Big\} + \int_0^T \overline{\Psi}^T(i\dot{\Psi} + A_0\Psi)\ , \tag{3.4.1}$$

where

$$\begin{aligned} D_0 X^b &:= \dot{X}^b + i[X^b, A_0]_M \\ &= \dot{X}^b - i[A_0, X^b]_M \end{aligned} \tag{3.4.2}$$

is a covariant derivative. Here we view Ψ as a column vector and $\overline{\Psi}^T$ denotes the row vector whose elements are the complex conjugates of the elements of Ψ. In addition, e and B are the same charge and constant magnetic field from Sec. 3.3, $\tilde{\omega}$ is a frequency (the term with $\tilde{\omega}$ is a quadratic potential for the noncommutative coordinates X^a), and θ is a parameter with units of length squared. We assume periodic boundary conditions on all the fields in the time direction, for example $X^a(0) = X^a(T)$, so that the time direction is a circle of circumference T. Note that the action as written here differs slightly in the details (signs, etc.) from Ref. [29], but is consistent with our interpretation of this model and the NCCS theory as describing a noncommutative fluid of particles with charge $-e < 0$.

At this point we would like to emphasize that the frequency $\tilde{\omega}$ appearing in the parabolic potential term of the CSMM has no relation to the cyclotron frequency $\omega_c = \frac{eB}{M}$ in the quantum Hall problem. Indeed, as we discussed in Sec. 3.3, the NCCS theory (and therefore the CSMM as well) describes a charged fluid in a magnetic field in the limit in which the mass M of the particles making up the fluid has been sent to zero. This sends the cyclotron frequency ω_c

to infinity. Therefore, the CSMM contains no information related to the cyclotron frequency or the energy of a Landau level.

We now discuss the gauge symmetry in the CSMM. If we ignore the term with $2\theta A_0$ for a moment, then we can see that the rest of the action is invariant under a $U(N)$ gauge transformation

$$X^a \quad \to \quad VX^a\overline{V}^T \tag{3.4.3a}$$

$$A_0 \quad \to \quad VA_0\overline{V}^T + iV\dot{\overline{V}}^T \tag{3.4.3b}$$

$$\Psi \quad \to \quad V\Psi\ , \tag{3.4.3c}$$

where $V(t)$ is an arbitrary time-dependent $U(N)$ matrix. The presence of the term $2\theta A_0$ in the action means that the action is not invariant under large $U(N)$ gauge transformations which are maps from $[0, T) \to U(N)$ which correspond to a nontrivial element in the homotopy group $\pi_1(U(N)) = \mathbb{Z}$. Since we would like $e^{i\frac{S_{CSMM}}{\hbar}}$ to be invariant under any gauge transformation, these large gauge transformations enforce a quantization rule on θ (the argument is identical to the argument for the full NCCS theory from Sec. 3.3) which states that

$$eB\theta = \hbar m,\ m \in \mathbb{Z}\ , \tag{3.4.4}$$

or

$$\theta = \ell_B^2 m\ ,\ m \in \mathbb{Z}\ . \tag{3.4.5}$$

The gauge field A_0 can be interpreted as a matrix Lagrange multiplier. If we look at the equation of motion resulting from a variation of A_0, then we find that A_0 enforces the constraint

$$ieB[X^1, X^2]_M + eB\theta\mathbb{I} - \Psi\overline{\Psi}^T = 0\ . \tag{3.4.6}$$

This constraint should be compared with Eq. (3.3.14) for the NCCS theory. In the NCCS case the contribution from the vector Ψ is absent and the constraint can only be realized by infinite-dimensional matrices (i.e., operators on $\mathcal{H}_F$). It is the presence of the vector Ψ which allows this constraint to be realized by finite-dimensional matrices, and this is why the CSMM can be thought of as a regularization of the NCCS theory. We refer the reader to Ref. [29] for the detailed analysis of the constraint in the classical solution of the CSMM (which is also closely related to the Calogero model of interacting particles in one spatial dimension). In this Chapter our main focus is on the solution of the model in the quantum case.

We now make a few remarks and set up some notation relating to the transformation properties of the fields under the action of the group $U(N)$. The field Ψ transforms in the fundamental representation of $U(N)$. We indicate this by writing the components of Ψ with an upper Latin index, Ψ^j, $j = 1, \ldots, N$. Under a $U(N)$ transformation we have

$$\Psi^j \to V^j{}_k \Psi^k \ , \tag{3.4.7}$$

where $V^j{}_k$ are the matrix elements of a unitary matrix V in $U(N)$. Next, the transpose conjugate $\overline{\Psi}^T$ transforms in the anti-fundamental representation of $U(N)$, $\overline{\Psi}^T \to \overline{\Psi}^T \overline{V}^T$. We indicate this by writing the components of $\overline{\Psi}^T$ with a lower index, $\overline{\Psi}_j$, $j = 1, \ldots, N$ (and recall that the components of $\overline{\Psi}^T$ are just the complex conjugates of the components of Ψ). In components we have

$$\overline{\Psi}_j \to \overline{\Psi}_k (\overline{V}^T)^k{}_j \ . \tag{3.4.8}$$

Finally, the matrix variables X^a transform in the adjoint representation of $U(N)$, $X^a \to V X^a \overline{V}^T$. Thus, the index structure of X^a is such that it has one upper and one lower index, $(X^a)^j{}_k$, $j, k = 1, \ldots, N$. The component form of the $U(N)$ transformation is then

$$(X^a)^j{}_k \to V^j{}_\ell (X^a)^\ell{}_m (\overline{V}^T)^m{}_k \ . \tag{3.4.9}$$

These conventions will be extremely useful later when we try to write down quantum states that respect the constraint of the CSMM.

We already mentioned that the matrix variables X^a are Hermitian matrices. Thus, their matrix elements $(X^a)^j{}_k$ are generically complex numbers. For the quantization of this system it will be more convenient to parametrize X^a in terms of scalar variables which are manifestly real. Then, when we quantize the theory, these real variables will be promoted to Hermitian operators on the quantum Hilbert space. Our choice of parametrization is as follows. First, let T^A, $A = 1, \ldots, N^2 - 1$, be the $N \times N$ generators, in the fundamental representation, of the Lie algebra of $SU(N)$. The matrices T^A are all Hermitian and traceless, and can be normalized to obey the relations

$$\mathrm{Tr}\{T^A T^B\} = \delta^{AB} \tag{3.4.10a}$$

$$\left[T^A, T^B\right]_M = i \sum_{C=1}^{N^2-1} f^{ABC} T^C \ , \tag{3.4.10b}$$

where f^{ABC} are the structure constants for $SU(N)$. These structure constants have a very important property which is that they are antisymmetric under exchange of *any* two indices A, B, or C (typically one only expects antisymmetry under $A \leftrightarrow B$). We will take advantage of this property later on. Using the generators T^A, we can parametrize X^a (for $a = 1, 2$) as

$$X^a(t) = x_0^a(t) \frac{\mathbb{I}}{\sqrt{N}} + \sum_{A=1}^{N^2-1} x_A^a(t) T^A \ , \tag{3.4.11}$$

where $x_0^a(t)$ and $x_A^a(t)$, $A = 1, \dots, N^2 - 1$, are N^2 real scalar variables. In the quantum theory these variables will be promoted to Hermitian operators. The factor of $\sqrt{N}$ on the identity matrix term was chosen for convenience.

The Poisson brackets for this system can be obtained from the corresponding symplectic form, which can in turn be read off from the action (which is first order in time derivatives). The full symplectic form on the phase space for this system is

$$\Omega = \Omega_X + \Omega_\Psi \tag{3.4.12}$$

with

$$\Omega_X = -eB \sum_{A=0}^{N^2-1} dx_A^1 \wedge dx_A^2 \tag{3.4.13}$$

and

$$\Omega_\Psi = -i\, d\Psi^j \wedge d\overline{\Psi}_j \ . \tag{3.4.14}$$

Our conventions for Poisson brackets are as follows. To any function f on phase space we associate a vector field $\underline{v}_f$ defined as the solution to the equation $df = -i_{\underline{v}_f}\Omega$. Then the Poisson bracket of any two functions f and g is given by $\{f, g\} = i_{\underline{v}_f} i_{\underline{v}_g}\Omega$. Using this convention we obtain the classical Poisson brackets (with $A, B = 0, \dots, N^2 - 1$ now)[8]

$$\begin{aligned} \{x_A^1, x_B^2\} &= \frac{1}{eB}\delta_{AB} && (3.4.15) \\ \{\Psi^j, \overline{\Psi}_k\} &= -i\delta_k^j \ . && (3.4.16) \end{aligned}$$

Upon quantization, in which we replace Poisson brackets with commutators as $\{f, g\} \to -\frac{i}{\hbar}[f, g]$, we find the commutation relations in the quantum CSMM to be

$$\begin{aligned} [x_A^1, x_B^2] &= i\ell_B^2 \delta_{AB} && (3.4.17a) \\ \left[\Psi^j, \overline{\Psi}_k\right] &= \hbar\delta_k^j \ , && (3.4.17b) \end{aligned}$$

where $\ell_B^2 = \frac{\hbar}{eB}$ is the magnetic length.

Finally, when the gauge field A_0 is set to zero, the Hamiltonian for this system is given by

$$H_{CSMM} = \frac{eB\tilde{\omega}}{2}\mathrm{Tr}\{\delta_{ab} X^a X^b\} \ . \tag{3.4.18}$$

All of the energy in the system is associated with the harmonic trap, and the only energy scale is associated with

[8]The reader should beware that the symbol B is now being used for two purposes. It is the strength of the magnetic field felt by the noncommutative fluid described by the CSMM and NCCS theory, and it is also (along with the capital Latin letters $A, C, \dots$) an index on the $SU(N)$ generators T^A and the variables x_A. It should be clear from the context whether B represents the magnetic field strength or an index.

frequency $\tilde{\omega}$ of the harmonic trap.

We now review the quantization of this model.

3.4.3 Quantization of the CSMM

We now discuss the quantization of the CSMM. Instead of trying to solve the constraint before quantization, we follow previous approaches to this model and first quantize, then impose the constraint on quantum states, i.e., physical states should be annihilated by the constraint operator. As we discussed above, upon quantization the matrix elements of X^1 and X^2 and the components of Ψ obey the quantum commutation relations from Eq. (3.4.17). In what follows we instead work with the oscillator variables

$$b^j = \frac{1}{\sqrt{\hbar}}\Psi^j \; , \tag{3.4.19}$$

with $b_j^\dagger = \frac{1}{\sqrt{\hbar}}\overline{\Psi}_j$, and

$$a_A = \frac{1}{\ell_B\sqrt{2}}(x_A^1 + i x_A^2) \; , \tag{3.4.20}$$

with $a_A^\dagger = \frac{1}{\ell_B\sqrt{2}}(x_A^1 - i x_A^2)$. These variables obey the commutation relations

$$\left[a_A, a_B^\dagger\right] = \delta_{AB} \tag{3.4.21}$$

$$\left[b^j, b_k^\dagger\right] = \delta^j_k \; . \tag{3.4.22}$$

The Hamiltonian for this system has the form

$$\begin{aligned} H_{CSMM} &= \frac{eB\tilde{\omega}}{2}\delta_{ab}(X^a)^j{}_k(X^b)^k{}_j \\ &= \frac{eB\tilde{\omega}}{2}\sum_{A=0}^{N^2-1}\delta_{ab}x_A^a x_A^b \; . \end{aligned} \tag{3.4.23}$$

In terms of the oscillator variables a_A and $a_A^\dagger$ this becomes

$$H_{CSMM} = \hbar\tilde{\omega}\frac{N^2}{2} + \hbar\tilde{\omega}\sum_{A=0}^{N^2-1}a_A^\dagger a_A \; . \tag{3.4.24}$$

Note that the first term represents the zero point energy of N^2 harmonic oscillators.

Next we turn to an analysis of the constraint. Classically, and in terms of the variables x_A^a, the constraint from Eq. (3.4.6) takes the form

$$-eB\sum_{A,B,C=1}^{N^2-1}x_A^1 x_B^2 f^{ABC}T^C + eB\theta\mathbb{I} - \Psi\overline{\Psi}^T = 0 \; . \tag{3.4.25}$$

To interpret the constraint in the quantum theory we study its j,k matrix element

$$-eB\sum_{A,B,C=1}^{N^2-1} x^1_A x^2_B f^{ABC}(T^C)^j{}_k + eB\theta\delta^j_k - \Psi^j\overline{\Psi}_k = 0\ . \tag{3.4.26}$$

In terms of the oscillator variables one can show that this matrix element of the constraint takes the form

$$i\frac{\hbar}{2}\sum_{A,B,C=1}^{N^2-1}(a^\dagger_A a_B + a_B a^\dagger_A) f^{ABC}(T^C)^j{}_k + eB\theta\delta^j_k - \hbar b^j b^\dagger_k = 0\ . \tag{3.4.27}$$

Note that in deriving this expression we needed to use the antisymmetry of the structure constants f^{ABC} under exchange of any of its indices. Finally, we use the commutation relations of the oscillator variables to rewrite this as

$$i\hbar\sum_{A,B,C=1}^{N^2-1} a^\dagger_A a_B f^{ABC}(T^C)^j{}_k + (eB\theta - \hbar)\delta^j_k - \hbar b^\dagger_k b^j = 0\ , \tag{3.4.28}$$

where we used the fact that $\sum_{A,B=1}^{N^2-1}\delta_{AB}f^{ABC} = 0$. Note the shift in the coefficient of the δ^j_k term which resulted from this manipulation[9]. Finally, we define $G^j{}_k$ to be the j,k matrix element of the constraint, but divided by a factor of $\hbar$ for convenience,

$$G^j{}_k = i\sum_{A,B,C=1}^{N^2-1} a^\dagger_A a_B f^{ABC}(T^C)^j{}_k + \left(\frac{\theta}{\ell_B^2} - 1\right)\delta^j_k - b^\dagger_k b^j\ . \tag{3.4.29}$$

In the quantum theory physical states $|\psi\rangle$ will be those states which satisfy

$$G^j{}_k|\psi\rangle = 0\ ,\ \forall\, j,k\ . \tag{3.4.30}$$

To understand the form of the physical states $|\psi\rangle$ we now analyze the constraint. First set $j = k$ and sum over all j. Then the constraint implies that

$$b^\dagger_j b^j|\psi\rangle = N\left(\frac{\theta}{\ell_B^2} - 1\right)|\psi\rangle\ . \tag{3.4.31}$$

Now we already know that θ is quantized as an integer, $\theta = \ell_B^2 m$, $m \in \mathbb{Z}$. If we take $m > 0$, then this equation reads as

$$b^\dagger_j b^j|\psi\rangle = N(m-1)|\psi\rangle\ . \tag{3.4.32}$$

[9]In Ref. [29] Polychronakos instead performs normal-ordering of the constraint by making the *replacement* $b^j b^\dagger_k \to b^\dagger_k b^j$. There is then no shift of the coefficient of the δ^j_k term. This difference between normal-ordering the constraint vs. treating it *as is* completely accounts for the fact that Polychronakos found that the CSMM with $\theta = \ell_B^2 m$ describes the $\nu = \frac{1}{m+1}$ Laughlin state, while we will find that it describes the $\nu = \frac{1}{m}$ Laughlin state (if we treated the constraint like Polychronakos then this would result in a trivial replacement of $m \to m+1$ in all results in this Chapter). Our treatment of the constraint is also identical to the treatment in Ref. [131], which discusses new Chern-Simons matrix models which can describe non-Abelian FQH states (our m is equal to their $k+1$ for their model with $p = 1$).

Thus, we find that the total number of b^j quanta in physical states must be equal to $N(m-1)$.

Next, we consider the off-diagonal components of the constraint. For this it is convenient to instead consider

$$G^A := G^j{}_k (T^A)^k{}_j \ , \tag{3.4.33}$$

which is the trace of the product of the constraint matrix (with elements $G^j{}_k$) and a generator T^A of $SU(N)$. We find that these operators take the form

$$G^A = -i\left(\mathcal{O}_X(T^A) + \mathcal{O}_\Psi(T^A)\right) \ , \tag{3.4.34}$$

where $\mathcal{O}_X(T^A)$ and $\mathcal{O}_\Psi(T^A)$ are the quantum operators which generate the action of the $SU(N)$ generator T^A on the X^a and Ψ variables, respectively. We define these operators and demonstrate their properties in Appendix B.1. Thus, the set of constraints

$$G^A|\psi\rangle = 0 \ , \ A = 1, \dots, N^2-1 \tag{3.4.35}$$

simply expresses the fact that physical states must be singlets under the total $SU(N)$ action, as originally noted by Polychronakos [29].

To summarize, we find that the constraint in the CSMM breaks up into two separate parts. The first is associated with the $U(1)$ part of the total $U(N)$ action and states that physical states $|\psi\rangle$ obey Eq. (3.4.32). The second part is associated with the $SU(N)$ part of $U(N)$ and states that physical states should be singlets under the $SU(N)$ action. Now that we understand the constraint, we can write down a basis of physical states satisfying this constraint. To this end we introduce the matrix-valued operator[10]

$$A^\dagger = a_0^\dagger \frac{\mathbb{I}}{\sqrt{N}} + \sum_{B=1}^{N^2-1} a_B^\dagger T^B \tag{3.4.36}$$

with matrix elements

$$(A^\dagger)^j{}_k = a_0^\dagger \frac{1}{\sqrt{N}} \delta^j_k + \sum_{B=1}^{N^2-1} a_B^\dagger (T^B)^j{}_k \ . \tag{3.4.37}$$

Then, as was shown by Hellerman and Van Raamsdonk in Ref. [117], one possible basis for all physical states is given by states of the form

$$|\{c_1, \dots, c_N\}\rangle = \mathrm{Tr}[(A^\dagger)^N]^{c_N} \cdots \mathrm{Tr}[A^\dagger]^{c_1} |\psi_0\rangle \tag{3.4.38}$$

[10]Perhaps a more precise notation for this operator would be $A^\dagger = a_0^\dagger \otimes \frac{\mathbb{I}}{\sqrt{N}} + \sum_{B=1}^{N^2-1} a_B^\dagger \otimes T^B$, which expresses the fact that $A^\dagger$ acts on the tensor product $\mathcal{H}_Q \otimes \mathcal{H}_N$ of an infinite-dimensional Hilbert space $\mathcal{H}_Q$ which arises upon quantization of the model, and an N-dimensional vector space $\mathcal{H}_N$ on which the classical matrix variables X^a act.

where each $c_j \in \mathbb{N}$ for $j = 1, \dots, N$, and

$$|\psi_0\rangle = \left[\epsilon^{j_1 \cdots j_N} b^\dagger_{j_1} (b^\dagger A^\dagger)_{j_2} \cdots (b^\dagger (A^\dagger)^{N-1})_{j_N}\right]^{(m-1)} |0\rangle \ . \quad (3.4.39)$$

Note that all $U(N)$ indices j, k, etc. are contracted in these expressions, and so every operator present is a singlet under the $SU(N)$ action. The overall power of $m-1$ in $|\psi_0\rangle$ is required to satisfy the $U(1)$ part of the constraint coming from Eq. (3.4.32).

Since the Hamiltonian of the CSMM just counts the total number of a_A quanta in a state, we find that $|\psi_0\rangle$ is the unique ground state of the CSMM, and that it has an energy

$$\begin{aligned} E_0 &= \hbar\tilde{\omega}\left[\frac{N^2}{2} + \frac{1}{2}(m-1)N(N-1)\right] \\ &= \hbar\tilde{\omega}\left[\frac{1}{2}mN^2 + \left(\frac{1-m}{2}\right)N\right] \ . \end{aligned} \quad (3.4.40)$$

The excited states $|\{c_1, \dots, c_N\}\rangle$ then have an energy

$$E(\{c_1, \dots, c_N\}) = E_0 + \hbar\tilde{\omega}\sum_{j=1}^{N} c_j j \ . \quad (3.4.41)$$

It follows that the partition function of the CSMM at an inverse temperature β is just

$$Z = \text{Tr}_Q[e^{-\beta H_{CSMM}}] = q^{\frac{1}{2}mN^2 + \left(\frac{1-m}{2}\right)N} \prod_{j=1}^{N} \frac{1}{1-q^j} \ , \quad (3.4.42)$$

where $\text{Tr}_Q[\cdot]$ denotes a trace over the quantum Hilbert space (consisting of physical states obeying the constraint of the CSMM), and where we defined $q = e^{-\beta\hbar\tilde{\omega}}$. As $N \to \infty$ the product $\prod_{j=1}^{N} \frac{1}{1-q^j}$ becomes the partition function for the oscillator modes of a single chiral boson, which we know is the edge theory of a Laughlin fractional quantum Hall state.

3.4.4 Density of the droplet

Here we review the calculation of the density of the FQH droplet described by the CSMM in the large N limit. We will see from this calculation that the CSMM with $\theta = \ell_B^2 m$ corresponds to the Laughlin state at filling fraction $\nu = \frac{1}{m}$. We do not find $\nu = \frac{1}{m+1}$ as we treated the constraint of Eq. (3.4.6) *as is* instead of normal-ordering it as in Polychronakos' original paper [29].

We compute the density of the droplet following the reasoning outlined by Polychronakos [29]. The key is to examine the eigenvalue of the operator

$$\text{Tr}\left\{\delta_{ab}X^aX^b\right\} = \sum_{A=0}^{N^2-1} \delta_{ab}x_A^a x_A^b \tag{3.4.43}$$

in the ground state $|\psi_0\rangle$ of the CSMM (the trace here is a matrix trace). Since this operator is proportional to H_{CSMM} we have Tr $\left\{\delta_{ab}X^aX^b\right\}|\psi_0\rangle = R^2|\psi_0\rangle$ where the eigenvalue R^2 is given by

$$R^2 = 2\ell_B^2\left(m\frac{N(N-1)}{2} + \frac{N}{2}\right) \ . \tag{3.4.44}$$

We interpret this eigenvalue as a sum of contributions from N different particles at different radial positions by writing it as

$$R^2 = \sum_{j=1}^{N} R_j^2 \ , \tag{3.4.45}$$

where

$$R_j^2 = 2\ell_B^2\left(m(j-1) + \frac{1}{2}\right) \ . \tag{3.4.46}$$

Indeed, the R_j^2 can be thought of as the eigenvalues of the classical matrix $\delta_{ab}X^aX^b$, since the operator R^2 is equal to the trace of this matrix. Thus, we think of the ground state of the droplet as containing N particles at definite radial positions R_j but with complete uncertainty in their angular position. In addition, since R_j^2 is linear in j, the area $\pi(R_j^2 - R_{j-1}^2) = 2\pi\ell_B^2 m$ of the annulus between consecutive particles is independent of j. This implies that the particles are distributed uniformly, i.e., the density is a constant within the droplet.

The size of the droplet is given by the largest value of R_j^2, which is

$$R_N^2 = 2\ell_B^2\left(m(N-1) + \frac{1}{2}\right) \approx 2\ell_B^2 mN \tag{3.4.47}$$

for large N. Then at large N we compute the density as being that of N particles evenly spread out over a disk of radius $R_N^2 \approx 2\ell_B^2 mN$, and so

$$\rho_0 = \frac{N}{\pi R_N^2} \approx \frac{1}{2\pi\ell_B^2 m} \ , \tag{3.4.48}$$

which is exactly the density of the Laughlin state with filling fraction $\nu = \frac{1}{m}$ (in the limit of a large number N of electrons).

3.5 Hall viscosity of the CSMM

We now compute the Hall viscosity in the CSMM following the calculation of Park and Haldane [101] (which we reviewed in Sec. 3.2). We find that the Hall viscosity tensor contains only a *single* contribution, and that this contribution is equal to the guiding center Hall viscosity of the Laughlin state. In other words, the CSMM lacks the Landau orbit contribution to the Hall viscosity, but does contain the (physically important) guiding center contribution.

To compute the Hall viscosity in this system we recall that in the fluid interpretation of the NCCS theory and the CSMM (which we reviewed at the end of Sec. 3.3), the variables X^a represent a noncommutative analogue of fluid coordinates in a Lagrange description of a fluid [27, 29, 129]. In the case of the CSMM, this is a finite droplet of noncommutative fluid. Thus, to compute the Hall viscosity we first need to identify the quantum operators Λ^{ab} which generate APDs (or strains) of the noncommutative fluid coordinates X^a. Since we expand the noncommutative coordinates in terms of the scalar variables x^a_A, $A = 0, \dots, N^2 - 1$, we can instead search for operators which implement APDs of these variables. These operators will then automatically implement the correct transformations of the X^a coordinates, as the operators do not act on the matrix indices of the X^a variables.

Since the commutation relations of the variables x^a_A are identical to the commutation relations of the guiding center coordinates in the quantum Hall problem, we immediately see that the desired operators are given by

$$\Lambda^{ab} = \frac{1}{4\ell_B^2} \sum_{A=0}^{N^2-1} \{x^a_A, x^b_A\} \ . \tag{3.5.1}$$

These operators obey the same algebra as in Eq. (3.2.5a). It follows that the unitary operators which implement the APDs are $U(\alpha) = e^{i\alpha_{ab}\Lambda^{ab}}$, with α_{ab} a constant symmetric matrix. To first order in α_{ab} we have (for all $A = 0, \dots, N^2 - 1$)

$$U(\alpha) x^a_A U(\alpha)^\dagger = x^a_A + \epsilon^{ab}\alpha_{bc} x^c_A + \cdots \tag{3.5.2}$$

which implies (for all j, k)

$$U(\alpha)(X^a)^j{}_k U(\alpha)^\dagger = (X^a)^j{}_k + \epsilon^{ab}\alpha_{bc}(X^c)^j{}_k + \cdots \tag{3.5.3}$$

It is important to note that the APD generators Λ^{ab} act only on the physical position indices a of the variables X^a. There is no action at all on the $U(N)$ indices j, k of the matrix elements $(X^a)^j{}_k$. Thus, the generators Λ^{ab} act identically on all matrix elements of X^a, and so they are indeed the correct quantum generators of APDs of the noncommutative fluid coordinates X^a (which we recall are actually $N \times N$ Hermitian matrices in the classical theory).

Now we want to compute the Hall viscosity in the ground state $|\psi_0\rangle$ of the CSMM. We compute this using a Kubo formula approach similar to that of Ref. [92]. We present the Kubo formula calculation of the Hall viscosity in

Appendix B.2. Our result is that the Hall viscosity tensor in this model takes the form (A is the area of the droplet)

$$\eta^{abcd}_{\text{CSMM}} = \frac{i\hbar}{A}\langle\psi_0|[\Lambda^{ab},\Lambda^{cd}]|\psi_0\rangle \ . \tag{3.5.4}$$

We note that the tensor $\eta^{abcd}_{\text{CSMM}}$ contains only a single contribution, as opposed to the two separate terms (guiding center and Landau orbit contributions) appearing in the discussion of the Hall viscosity tensor from Sec. 3.2. Note that in deriving this result it was crucial that the CSMM has a unique ground state and a finite energy gap set by the frequency $\tilde{\omega}$ of the harmonic trap.

Due to the commutation relations of the generators Λ^{ab} (which are the same as Eq. (3.2.5a)), the four index tensor $\eta^{abcd}_{\text{CSMM}}$ can again be expressed in terms of a symmetric two-index tensor

$$\eta^{ab}_{\text{CSMM}} = -\frac{\hbar}{A}\langle\psi_0|\Lambda^{ab}|\psi_0\rangle \ . \tag{3.5.5}$$

Therefore, to compute the Hall viscosity tensor of the CSMM, we just need to compute the expectation values $\langle\psi_0|\Lambda^{ab}|\psi_0\rangle$. To compute these we first note that the CSMM Hamiltonian can be written as

$$H_{CSMM} = \hbar\tilde{\omega}\delta_{ab}\Lambda^{ab} = \hbar\tilde{\omega}(\Lambda^{11}+\Lambda^{22}) \ . \tag{3.5.6}$$

From this we can already deduce that

$$\langle\psi_0|\delta_{ab}\Lambda^{ab}|\psi_0\rangle = \frac{E_0}{\hbar\tilde{\omega}} = \frac{1}{2}mN^2 + \left(\frac{1-m}{2}\right)N \ . \tag{3.5.7}$$

We can go further and compute the individual expectation values of Λ^{11} and Λ^{22} by deriving a Virial theorem for the CSMM. To derive this theorem consider the operator

$$Q = \sum_{A=0}^{N^2-1} x^1_A x^2_A \ . \tag{3.5.8}$$

A short computation shows that

$$[Q,\delta_{ab}\Lambda^{ab}] = 2i\ell_B^2(-\Lambda^{11}+\Lambda^{22}) \ . \tag{3.5.9}$$

If we take the expectation value of this equation in the state $|\psi_0\rangle$ (or any eigenstate of $\delta_{ab}\Lambda^{ab}$), then we find that

$$\langle\psi_0|\Lambda^{11}|\psi_0\rangle = \langle\psi_0|\Lambda^{22}|\psi_0\rangle \ . \tag{3.5.10}$$

Combining this result with Eq. (3.5.7) gives the result that

$$\langle\psi_0|\Lambda^{11}|\psi_0\rangle = \langle\psi_0|\Lambda^{22}|\psi_0\rangle = \frac{1}{2}\left[\frac{1}{2}mN^2 + \left(\frac{1-m}{2}\right)N\right] . \tag{3.5.11}$$

Finally, it remains to compute the expectation value of the off-diagonal generator $\Lambda^{12} = \Lambda^{21}$. In terms of the oscillator variables a_A and $a^\dagger_A$ this operator takes the form

$$\Lambda^{12} = \frac{1}{4i}\sum_{A=0}^{N^2-1}\left(a_A a_A - a^\dagger_A a^\dagger_A\right) . \tag{3.5.12}$$

Now all eigenstates of H_{CSMM} are eigenstates of the total number operator for the a_A oscillators. Since Λ^{12} clearly does not commute with the total number operator, we immediately conclude that the expectation value of Λ^{12} in any eigenstate of H_{CSMM} is zero.

Therefore our final result for the expectation value of the APD generators Λ^{ab} in the CSMM ground state is

$$\langle\psi_0|\Lambda^{ab}|\psi_0\rangle = \frac{1}{2}\left[\frac{1}{2}mN^2 + \left(\frac{1-m}{2}\right)N\right]\delta^{ab} . \tag{3.5.13}$$

This means that we can write $\eta^{ab}_{\text{CSMM}} = \eta_{\text{CSMM}}\delta^{ab}$, where the coefficient η_{CSMM} of Hall viscosity in this model is equal to

$$\eta_{\text{CSMM}} = -\frac{\hbar}{A}\frac{1}{2}\left[\frac{1}{2}mN^2 + \left(\frac{1-m}{2}\right)N\right] . \tag{3.5.14}$$

Now since $A = \pi R_N^2 \approx 2\pi\ell_B^2 mN$ for the CSMM at large N, this exactly matches the result (before regularization) for the *guiding center Hall viscosity* η_H of the $\nu = \frac{1}{m}$ Laughlin state. The Landau orbit contribution $\tilde{\eta}_H$ is absent in the CSMM. Finally, as was the case for the ordinary Laughlin state, this result can be regularized by subtracting off the extensive term in η_{CSMM} (or the superextensive term in $\langle\psi_0|\Lambda^{ab}|\psi_0\rangle$). We discuss a fluid interpretation of this regularization of the Hall viscosity later in Sec. 3.7.

3.6 Hall conductance of the CSMM in a non-uniform electric field

In this section we study the Hall conductance of the CSMM when it is subjected to a non-uniform electric field. Our motivation for studying this setup is the well-known result of Hoyos and Son which shows that in a quantum Hall state the Hall conductance $\sigma_H(\mathbf{k})$ at finite wave vector $\mathbf{k}$ has a universal contribution at order k^2 ($k^2 = \delta^{ab}k_a k_b$) which is related to the Hall viscosity [93] (see also Ref. [92] for a Kubo formula approach to this relation). We find a similar contribution in the CSMM, but depending only on the guiding center Hall viscosity as opposed to the full Hall viscosity. Again, this is not surprising as we only expect the CSMM to describe the dynamics of the guiding center degrees of freedom in a FQH state.

In this section we first review the result of Ref. [93] on the Hall conductance at finite wave vector. We then warm up by calculating the Hall conductance of the CSMM subjected to a *uniform* electric field. The reason for this is that there are several subtle points associated with the computation of the Hall conductance in the CSMM that we want to explain clearly. Finally, we compute the Hall conductance of the CSMM in a non-uniform electric field, where we find a result which resembles the result of Hoyos and Son [93], but with the full Hall viscosity replaced by the guiding center Hall viscosity. We note here that the Hall conductance of the NCCS theory in a uniform electric field was computed previously in Refs. [120, 122] at the classical level by solving the equations of motion for the NCCS theory in a uniform electric field. We therefore emphasize that our treatment in this section deals directly with the quantized CSMM theory as opposed to the classical NCCS theory.

3.6.1 The result of Hoyos and Son

We start by reviewing the result of Ref. [93]. Consider a quantum Hall system in a non-uniform electric field $\mathbf{E} = (E(\mathbf{x}), 0)$ pointing in the x^1 direction, and where the spatial dependence is only on the x^1 coordinate, so that $\partial_2 E(\mathbf{x}) = 0$. The Hall conductance $\sigma_H(\mathbf{k})$ at finite wave vector is defined by the relation

$$j^2(\mathbf{k}) = \sigma_H(\mathbf{k}) E(\mathbf{k}) \ , \tag{3.6.1}$$

where $j^2(\mathbf{k})$ is the Fourier transform of the charge current in the x^2 direction, and $E(\mathbf{k})$ is the Fourier transform of $E(\mathbf{x})$. The result of Ref. [93] is that (recall that $E(\mathbf{x})$ is a function of only x^1)

$$\frac{\sigma_H(\mathbf{k})}{\sigma_H(\mathbf{0})} = 1 + C_2 (k_1 \ell_B)^2 + \cdots \ , \tag{3.6.2}$$

where the Hall conductance at zero wave vector is simply (ν is the filling fraction)

$$\sigma_H(\mathbf{0}) = \nu \frac{e^2}{h} \ . \tag{3.6.3}$$

The coefficient C_2 is given by

$$C_2 = \frac{\eta_{tot}}{\hbar \rho_0} - \frac{2\pi}{\nu} \frac{\ell_B^2}{\hbar \omega_c} B^2 \mathcal{E}''(B) \ , \tag{3.6.4}$$

where η_{tot} denotes the full Hall viscosity of the quantum Hall state (as opposed to just the guiding center part), $\mathcal{E}(B)$ is the energy density of the quantum Hall state viewed as a function of the external field B, and $\mathcal{E}''(B)$ denotes the second derivative of $\mathcal{E}(B)$ with respect to B. In addition, ρ_0 denotes the number density of the quantum Hall state, and $\omega_c = \frac{eB}{M}$ is the cyclotron frequency, where M is the mass of the particles making up the quantum Hall state. As an example, for a quantum Hall state consisting of N electrons in the lowest Landau level and occupying an area A, we have $\mathcal{E}(B) = \frac{\hbar\omega_c}{2}\frac{N}{A} = \frac{\hbar\omega_c}{2}\rho_0$, and for a Laughlin $\nu = \frac{1}{m}$ state this gives $\mathcal{E}(B) = \frac{\hbar\omega_c}{4\pi\ell_B^2 m}$.

In the context of the CSMM, the quantity that we actually compute is the current at the location of the center of mass of the droplet (we explain the reason for this in the next subsection). Therefore we need to Fourier transform the result of Hoyos and Son back to real space in order to compare with our calculation in the CSMM later in this section. In real space we find that

$$j^2(\mathbf{x}) = \nu\frac{e^2}{h}\left(E(\mathbf{x}) - C_2\ell_B^2\partial_1^2 E(\mathbf{x}) + \dots\right) . \tag{3.6.5}$$

In particular, at the origin $\mathbf{x} = \mathbf{0}$ (where the center of mass of a uniform droplet would be located) we have

$$j^2(\mathbf{x} = \mathbf{0}) = \nu\frac{e^2}{h}\left(E^{(0)} - C_2\ell_B^2 E^{(2)} + \dots\right) , \tag{3.6.6}$$

where $E^{(0)}$ and $E^{(2)}$ are the coefficients in the Taylor series expansion of $E(\mathbf{x})$ about the origin,

$$E(\mathbf{x}) = E^{(0)} + E^{(1)}x^1 + \frac{1}{2!}E^{(2)}(x^1)^2 + \dots , \tag{3.6.7}$$

and where we again remind the reader that we assumed that $E(\mathbf{x})$ has no x^2 dependence.

3.6.2 Uniform electric field

We now compute the Hall conductance of the CSMM in a uniform electric field. Our reason for treating this simple case first is to highlight a few subtleties in the calculation of the Hall conductance of the CSMM. The first subtlety is associated with the fact that one cannot resolve individual points in space in the CSMM, since the spatial coordinates are actually the noncommuting matrices X^1 and X^2. However, in the CSMM one can still define a notion of the center of mass coordinate of the FQH droplet, and the expectation value of this center of mass coordinate can be computed in any state $|\psi\rangle$ of the quantized CSMM. We define the center of mass coordinates X^a_{COM} as

$$X^a_{\text{COM}} = \frac{1}{N}\text{Tr}\{X^a\} = \frac{x_0^a}{\sqrt{N}} , \tag{3.6.8}$$

where in the second equality we evaluated the trace and found that X^a_{COM} is proportional to the variable x_0^a introduced in Eq. (3.4.11) of Sec. 3.4. To motivate this definition we simply note that if the X^a were diagonal matrices, then their diagonal elements could be interpreted as the positions of N particles, and then $\frac{1}{N}\text{Tr}\{X^a\}$ would agree with the usual definition of the center of mass coordinate of N particles (assuming all particles have equal masses).

Our strategy to compute the Hall conductance in the CSMM is to compute the drift velocity $\mathbf{v}_{drift}$ of the center of mass coordinate when the system is placed in an electric field $\mathbf{E}$. We can then use the fact that the CSMM describes a droplet of particles with charge $-e$ and density $\rho_0 = \frac{1}{2\pi\ell_B^2 m}$ (computed in Sec. 3.4) to compute the charge current $\mathbf{j}_{\text{COM}}$ at the center of mass as

$$\mathbf{j}_{\text{COM}} = -e\rho_0 \mathbf{v}_{drift} \ . \tag{3.6.9}$$

The result can then be compared with the result of Hoyos and Son for the current at the origin (location of the center of mass) as expressed in Eq. (3.6.6).

Next, we need to discuss the issue of how to couple the CSMM to an external electric field. This can be done using the fluid interpretation of this theory from Sec. 3.3. First, recall from Sec. 3.3 that an ordinary charged fluid on commutative flat space $\mathbb{R}^2$ can be coupled to a background electromagnetic field by including vector and scalar potentials $\mathcal{A}_a(t, \mathbf{X})$ and $\varphi(t, \mathbf{X})$, respectively, in the action for the Lagrange description of this fluid, Eq. (3.3.34). In our case we are only interested in adding a scalar potential $\varphi(t, \mathbf{X})$ for the external electric field. Using the fluid interpretation we can incorporate this potential into the NCCS theory by adding a term to the NCCS action of the form

$$S_{EM} = e\int_0^T dt\, \text{Tr}_{\mathcal{H}_F}\left\{\hat{\varphi}(\hat{\mathbf{X}}, t)\right\} \ , \tag{3.6.10}$$

where the operator $\hat{\varphi}(\hat{\mathbf{X}}, t)$ is the operator representing the scalar potential for the external electromagnetic field (and recall that the charge of the particles is $q = -e$).

In defining the operator $\hat{\varphi}(\hat{\mathbf{X}}, t)$ we encounter an ordering ambiguity. For example if the scalar potential for the electric field configuration on a commutative space is $\varphi(t, \mathbf{X}) = X^1X^2$, then we could define $\hat{\varphi}(\hat{\mathbf{X}}, t) = \hat{X}^1\hat{X}^2$, $\hat{\varphi}(\hat{\mathbf{X}}, t) = \hat{X}^2\hat{X}^1$, or the symmetric Weyl ordering $\hat{\varphi}(\hat{\mathbf{X}}, t) = \frac{1}{2}\left(\hat{X}^1\hat{X}^2 + \hat{X}^2\hat{X}^1\right)$, for example. We choose to use Weyl ordering since this is consistent with our use of Weyl ordering to go between star product and operator formulations of noncommutative field theory (recall Eq. (3.3.8)), however, in the examples of this section we do not actually encounter this ordering ambiguity. Weyl-ordering for the external field was also adopted by the authors of Ref. [120], who also considered the NCCS theory in the presence of external fields.

Finally, to couple the CSMM to the external electromagnetic field we use the same action S_{EM} as above but replace the operators $\hat{X}^a$ on the infinite-dimensional space $\mathcal{H}_F$ with the finite $N \times N$ matrix variables of the CSMM. From this action we can then read off the new Hamiltonian for the CSMM coupled to the external electric field.

There is one more subtlety with the calculation of the Hall conductance of the CSMM that we need to address before we can proceed. The issue is that the parabolic potential in the CSMM competes with the applied electric field to determine the long time behavior of the CSMM in the presence of the electric field. This is best illustrated for the case of the CSMM in a constant electric field $E^{(0)}$ pointing in the x^1 direction. The Hamiltonian describing this system is

$$\begin{aligned} H' &= H_{CSMM} + eE^{(0)}\mathrm{Tr}\{X^1\} \\ &= H_{CSMM} + eE^{(0)}NX^1_{\rm COM} , \end{aligned} \tag{3.6.11}$$

and where the trace is a classical matrix trace. To derive this Hamiltonian we used the fluid interpretation of the CSMM theory and incorporated a scalar potential $\varphi(t, \mathbf{X}) = -E^{(0)}X^1$ to describe the coupling to a constant electric field in the x^1 direction. This Hamiltonian can be immediately diagonalized by noting that

$$H' = T(\mathbf{R})H_{CSMM}T(\mathbf{R})^\dagger - \frac{eN(E^{(0)})^2}{2B\tilde{\omega}} , \tag{3.6.12}$$

where $T(\mathbf{R})$ is a unitary translation operator[11] (similar to a magnetic translation) of the form

$$T(\mathbf{R}) = \exp\left\{-\frac{i\epsilon_{ab}NX^a_{\rm COM}R^b}{\ell_B^2}\right\} , \tag{3.6.13}$$

and where in this case

$$\mathbf{R} = \left(\frac{E^{(0)}}{B\tilde{\omega}}, 0\right) . \tag{3.6.14}$$

The ground state of this Hamiltonian is $|\psi_0'\rangle = T(\mathbf{R})|\psi_0\rangle$ and represents a stationary state with $\langle\psi_0'|X^1_{\rm COM}|\psi_0'\rangle = -\frac{E^{(0)}}{B\tilde{\omega}}$ and $\langle\psi_0'|X^2_{\rm COM}|\psi_0'\rangle = 0$, which corresponds to the equilibrium position in the total potential

$$V = \frac{eBN\tilde{\omega}}{2}\delta_{ab}X^a_{\rm COM}X^b_{\rm COM} + eE^{(0)}NX^1_{\rm COM} \tag{3.6.15}$$

felt by the center of mass.

We see that if we simply diagonalize the Hamiltonian H' for the CSMM in the presence of the external field, we find no time dependence and, in the ground state, the center of mass of the droplet just sits at its equilibrium position $(-\frac{E^{(0)}}{B\tilde{\omega}}, 0)$ under the influence of the combined forces of the parabolic potential and the applied electric field.

To compute the Hall conductance of this model we instead need to consider a non-equilibrium situation in which we start with the system in the ground state $|\psi_0\rangle$ of the unperturbed CSMM (which we will now assume is properly normalized) and then suddenly turn on the electric field. We then study the time evolution of the center of mass coordinate at small times $t \ll \frac{1}{\tilde{\omega}}$, where $\frac{1}{\tilde{\omega}}$ is the time scale set by the parabolic potential. Therefore we consider the

[11] We have $[X^a_{\rm COM}, X^b_{\rm COM}] = \frac{i\ell_B^2}{N}\epsilon^{ab}$ and $T(\mathbf{R})X^a_{\rm COM}T(\mathbf{R})^\dagger = X^a_{\rm COM} + R^a$.

"quantum quench" problem in which the state of the system at time t is given by

$$|\psi(t)\rangle = e^{-i\frac{H't}{\hbar}}|\psi_0\rangle\ , \tag{3.6.16}$$

where $|\psi_0\rangle$ is the ground state of the unperturbed CSMM Hamiltonian H_{CSMM}, and H' is the perturbed CSMM Hamiltonian including the applied electric field. We then compute

$$\langle\psi(t)|X^a_{\text{COM}}|\psi(t)\rangle = \langle\psi_0|X^a_{\text{COM}}|\psi_0\rangle + \frac{it}{\hbar}\langle\psi_0|[H', X^a_{\text{COM}}]|\psi_0\rangle + \dots \tag{3.6.17}$$

and identify the drift velocity $\mathbf{v}_{drift}$ of the center of mass with the term linear in t in this expansion,

$$v^a_{drift} = \frac{i}{\hbar}\langle\psi_0|[H', X^a_{\text{COM}}]|\psi_0\rangle\ . \tag{3.6.18}$$

We now consider the case of a uniform electric field $E^{(0)}$ pointing in the x^1 direction so that H' takes the form shown in Eq. (3.6.11). In this case the drift velocity evaluates to

$$\mathbf{v}_{drift} = \left(0, -\frac{E^{(0)}}{B}\right)\ . \tag{3.6.19}$$

Then the non-zero part of the charge current at the center of mass of the droplet, at times $t \ll \frac{1}{\tilde{\omega}}$, is

$$\begin{aligned} j^2_{\text{COM}} &= e\rho_0\frac{E^{(0)}}{B} \\ &= \nu\frac{e^2}{h}E^{(0)}\ , \end{aligned} \tag{3.6.20}$$

with $\nu = \frac{1}{m}$, and where we used $\rho_0 = \frac{1}{2\pi\ell_B^2 m}$. Therefore we find that the Hall conductance of the CSMM with $\theta = \ell_B^2 m$ is given by

$$\sigma_H = \frac{1}{m}\frac{e^2}{h}\ , \tag{3.6.21}$$

exactly as in the $\nu = \frac{1}{m}$ Laughlin state.

For the case of a uniform electric field we can actually go further and compute the full time dependence of the center of mass coordinate. We find that

$$\langle\psi(t)|X^1_{\text{COM}}|\psi(t)\rangle = \frac{E^{(0)}}{B\tilde{\omega}}\left(-1+\cos(\tilde{\omega}t)\right) \tag{3.6.22a}$$

$$\langle\psi(t)|X^2_{\text{COM}}|\psi(t)\rangle = -\frac{E^{(0)}}{B\tilde{\omega}}\sin(\tilde{\omega}t)\ . \tag{3.6.22b}$$

We see that the center of mass moves in a large circle about its equilibrium position $(-\frac{E^{(0)}}{B\tilde{\omega}}, 0)$, but that at early times $t \ll \frac{1}{\tilde{\omega}}$ the droplet drifts in the x^2 direction with velocity vector $\mathbf{v}_{drift} = \left(0, -\frac{E^{(0)}}{B}\right)$.

3.6.3 Non-uniform electric field

We now compute the Hall conductance of the CSMM in a non-uniform electric field. We consider an electric field which points in the x^1 direction, and which depends only on the x^1 coordinate. Since we are interested in contributions to the current which depend on the second derivative of the electric field, it is sufficient to consider an electric field which depends at most quadratically on the x^1 coordinate. Thus, for an ordinary classical charged fluid described by the action of Eq. (3.3.34), we would add a scalar potential of the form

$$\varphi(t, \mathbf{X}) = -E^{(0)}X^1 - \frac{1}{2}E^{(1)}(X^1)^2 - \frac{1}{3!}E^{(2)}(X^1)^3 \ , \tag{3.6.23}$$

which corresponds, after computing minus the spatial gradient, to an electric field $\mathbf{E} = (E(\mathbf{X}), 0)$ with

$$E(\mathbf{X}) = E^{(0)} + E^{(1)}X^1 + \frac{1}{2}E^{(2)}(X^1)^2 \ . \tag{3.6.24}$$

The coefficients $E^{(j)}$, $j = 0, 1, 2$ in this expression (which are all fixed real numbers) can be understood as the coefficients in the Taylor expansion of $E(\mathbf{X})$ about the origin.

This form of the scalar potential for the ordinary classical fluid, combined with the considerations from earlier in this section on how to couple the CSMM to external fields, leads to a Hamiltonian

$$H' = H_{CSMM} + H_1 \tag{3.6.25}$$

with

$$H_1 = e\text{Tr}\left\{E^{(0)}X^1 + \frac{1}{2}E^{(1)}(X^1)^2 + \frac{1}{3!}E^{(2)}(X^1)^3\right\} \ , \tag{3.6.26}$$

where the trace denotes a matrix trace. This Hamiltonian then describes the CSMM in the presence of a non-uniform electric field in the x^1 direction. To compute the Hall conductance we again consider a time-dependent problem where the state at time t is given by $|\psi(t)\rangle = e^{-i\frac{H't}{\hbar}}|\psi_0\rangle$ with $|\psi_0\rangle$ the ground state of H_{CSMM}. The drift velocity is again given by Eq. (3.6.18) and since $\langle\psi_0|[H_{CSMM}, X^a_{\text{COM}}]|\psi_0\rangle = 0$ (since $|\psi_0\rangle$ is an eigenstate of H_{CSMM}), this reduces to

$$v^a_{drift} = \frac{i}{\hbar}\langle\psi_0|[H_1, X^a_{\text{COM}}]|\psi_0\rangle \ . \tag{3.6.27}$$

It remains to actually compute the matrix element $\langle\psi_0|[H_1, X^a_{\text{COM}}]|\psi_0\rangle$.

To compute this matrix element we first note that we already know the answer for the term in H_1 proportional to $E^{(0)}$ from the previous subsection. Next, we can immediately see that the term proportional to $E^{(1)}$ will vanish since the commutator of Tr$\{(X^1)^2\}$ with X^a_{COM} is linear in the center of mass coordinate and we know that $\langle\psi_0|X^a_{\text{COM}}|\psi_0\rangle = 0$ in the unperturbed ground state of the CSMM. To handle the term proportional to $E^{(2)}$ we use Eq. (3.4.11) to find that

$$\text{Tr}\{(X^1)^3\} = \frac{(x^1_0)^3}{\sqrt{N}} + \frac{3}{\sqrt{N}} x^1_0 \sum_{A=1}^{N^2-1} x^1_A x^1_A + \sum_{A,B,C=1}^{N^2-1} x^1_A x^1_B x^1_C \text{Tr}\{T^A T^B T^C\} \ . \tag{3.6.28}$$

Then we have $[\text{Tr}\{(X^1)^3\}, X^1_{\text{COM}}] = 0$ and

$$[\text{Tr}\{(X^1)^3\}, X^2_{\text{COM}}] = \frac{3i\ell_B^2}{N} \sum_{A=0}^{N^2-1} x^1_A x^1_A \ . \tag{3.6.29}$$

We find that $v^1_{drift} = 0$, while

$$\begin{aligned} v^2_{drift} &= -\frac{E^{(0)}}{B} + \frac{i}{\hbar}\left(e\frac{E^{(2)}}{3!}\right)\frac{3i\ell_B^2}{N}\langle\psi_0|\sum_{A=0}^{N^2-1} x^1_A x^1_A|\psi_0\rangle \\ &= -\frac{E^{(0)}}{B} - \frac{eE^{(2)}\ell_B^4}{N}\langle\psi_0|\Lambda^{11}|\psi_0\rangle \\ &= -\frac{E^{(0)}}{B} + \frac{E^{(2)}\ell_B^2}{B}\frac{\eta_{\text{CSMM}}}{\hbar\rho_0} \ , \end{aligned} \tag{3.6.30}$$

where we used the fact that $\langle\psi_0|\Lambda^{11}|\psi_0\rangle = -\frac{A}{\hbar}\eta_{\text{CSMM}}$ and $\rho_0 = \frac{N}{A}$. If we now compute $j^2_{\text{COM}} = -e\rho_0 v^2_{drift}$ then we find that

$$\begin{aligned} j^2_{\text{COM}} &= \nu\frac{e^2}{h}\left(E^{(0)} - E^{(2)}\ell_B^2\frac{\eta_{\text{CSMM}}}{\hbar\rho_0}\right) \\ &= \nu\frac{e^2}{h}\left(E^{(0)} - E^{(2)}\ell_B^2\frac{\eta_H}{\hbar\rho_0}\right) \ , \end{aligned} \tag{3.6.31}$$

where the second line follows from the fact that $\eta_{\text{CSMM}} = \eta_H$, where η_H was the guiding center Hall viscosity for the Laughlin state. Finally, we should regularize this expression to obtain a finite answer for the current in the $N \to \infty$ limit. This just amounts to the replacement $\eta_H \to \eta_{H,reg}$ in the final expression (we discuss the physical interpretation of this regularization in Sec. 3.7). Therefore our final expression for the center of mass current in a non-uniform electric field is

$$j^2_{\text{COM}} = \nu\frac{e^2}{h}\left(E^{(0)} - E^{(2)}\ell_B^2\frac{\eta_{H,reg}}{\hbar\rho_0}\right) \ . \tag{3.6.32}$$

Eq. (3.6.32) is the main result of this section.

It is interesting to compare Eq. (3.6.32) with the result of Hoyos and Son, Eq. (3.6.6), where the coefficient C_2 was given in Eq. (3.6.4). We see that the CSMM result contains a contribution like the first term in C_2, but with the total Hall viscosity η_{tot} replaced with the guiding center Hall viscosity $\eta_{H,reg}$. As we remarked earlier, this makes sense because we only expect the CSMM to describe the dynamics of the guiding center degrees of freedom in the quantum Hall problem. We also find that the CSMM result does not contain any contribution resembling the second term in C_2 which is proportional to $\mathcal{E}''(B)$. This is also not surprising since the CSMM itself does not contain any information about the energy associated with electrons filling a Landau level. Indeed, we can see from the fluid interpretation of the NCCS theory from Sec. 3.3 that the NCCS theory (and therefore the CSMM theory which is a regularization of it), is obtained by sending the energy scale $\hbar\omega_c$ to infinity. Therefore we find that the CSMM accurately captures the *guiding center contribution* to the response of a FQH state to a non-uniform electric field.

3.7 $N \to \infty$ limit, regularization of the Hall viscosity, and fluid interpretation

In Ref. [101] Park and Haldane argued that one should regularize the guiding center Hall viscosity by subtracting the extensive term in $\eta_H = -\frac{\hbar}{A}\frac{1}{2}\left[\frac{1}{2}mN^2 + \left(\frac{1-m}{2}\right)N\right]$, which amounts to subtracting the term $\frac{1}{2}mN^2$ from

$$\frac{1}{2}mN^2 + \left(\frac{1-m}{2}\right)N \, . \tag{3.7.1}$$

In the quantum Hall problem this regularization (or something similar to it) is necessary to obtain a finite value for the guiding center Hall viscosity in the thermodynamic limit $N \to \infty$.

In this section we give an interpretation of this regularization scheme in the context of the fluid interpretation (reviewed in the last subsection of Sec. 3.3) of the NCCS theory and CSMM. Our starting point is to note that the expectation value $\langle\psi_0|\Lambda^{ab}|\psi_0\rangle$ in the CSMM is actually proportional to the total angular momentum of the state $|\psi_0\rangle$. The fact that the Hall viscosity is related to angular momentum has been discussed extensively in Ref. [91], so this is not a new observation. However, this observation will allow us to understand the origin of the superextensive term $\frac{1}{2}mN^2$ in $\langle\psi_0|\Lambda^{ab}|\psi_0\rangle$, and to explain why it should be subtracted when computing the Hall viscosity of the CSMM.

We start by deriving an expression for the angular momentum in the CSMM theory. To do this we use the fluid interpretation of the NCCS theory and CSMM from the last part of Sec. 3.3. Our derivation of the expression for the angular momentum consists of several steps. First, we derive an expression for the angular momentum of a classical fluid of charged particles on a commutative space $\mathbb{R}^2$ and in the presence of a constant background magnetic field. Next, we take the limit in which the mass of the particles making up the fluid goes to zero. We then perform the noncommutative deformation of the expression for the angular momentum to obtain an expression for the angular momentum in NCCS theory. Finally, the expression for the angular momentum in NCCS theory can also be used for the CSMM, after we replace the infinite-dimensional operator variables in the NCCS theory with the $N \times N$ matrix variables of the CSMM.

We start with the action for a fluid of particles of mass M, charge $q = -e$, and constant (initial) density ρ_0 in a constant magnetic field B (see the discussion in the last subsection of Sec. 3.3),

$$S = \int_0^T dt \int d^2\mathbf{x}\, \rho_0 \left(\frac{1}{2} M \delta_{ab} \dot{X}^a \dot{X}^b - \frac{eB}{2} \epsilon_{ab} X^a \dot{X}^b \right) \ , \tag{3.7.2}$$

where we remind the reader that for the classical fluid the fields $X^a(t,\mathbf{x})$ are ordinary functions of time t and spatial coordinates $\mathbf{x} \in \mathbb{R}^2$. For now we omit the Lagrange multiplier field $A_0(t,\mathbf{x})$ which keeps the density fixed to ρ_0 at all times, as this term plays no role in the definition of the angular momentum of the theory. The momentum variables $P_a(t,\mathbf{x})$ canonically conjugate to $X^a(t,\mathbf{x})$ are obtained by differentiating the Lagrangian[12] with respect to $\dot{X}^a$, and we have

$$P_1 \ = \ M\dot{X}^1 + \frac{eB}{2} X^2 \tag{3.7.3}$$

$$P_2 \ = \ M\dot{X}^2 - \frac{eB}{2} X^1 \ . \tag{3.7.4}$$

The expression for the angular momentum of this fluid is then

$$\begin{aligned} L_z \ &= \ \int d^2\mathbf{x}\, \rho_0 \left(X^1 P_2 - X^2 P_1 \right) \\ &= \ \int d^2\mathbf{x}\, \rho_0 \left\{ M \epsilon_{ab} X^a \dot{X}^b - \frac{eB}{2} \delta_{ab} X^a X^b \right\} \ , \end{aligned} \tag{3.7.5}$$

and the limit $M \to 0$ gives

$$L_z = -\int d^2\mathbf{x}\, \rho_0 \frac{eB}{2} \delta_{ab} X^a X^b \ . \tag{3.7.6}$$

Next, we set $\rho_0 = \frac{1}{2\pi\theta}$ as is appropriate for the fluid interpretation of NCCS theory, and we perform the noncommutative deformation of this expression (see Sec. 3.3) by replacing $\frac{1}{2\pi\theta} \int d^2\mathbf{x}\, (\cdots) \to \text{Tr}_{\mathcal{H}_F} \{\cdots\}$ and $X^a(t,\mathbf{x}) \to \hat{X}^a(t)$. This gives an expression for the angular momentum in NCCS theory,

$$L_z = -\frac{eB}{2} \text{Tr}_{\mathcal{H}_F} \left\{ \delta_{ab} \hat{X}^a \hat{X}^b \right\} \ . \tag{3.7.7}$$

Finally, we obtain an expression for the angular momentum of the CSMM by replacing the operators $\hat{X}^a(t)$ with the $N \times N$ matrix variables $X^a(t)$ of the CSMM, and by replacing the trace over the infinite-dimensional space $\mathcal{H}_F$ by

[12] We define the Lagrangian $\mathcal{L}$ by $S = \int dt \int d^2\mathbf{x}\, \rho_0\, \mathcal{L}$.

the trace for $N \times N$ matrices,

$$L_{z,\text{CSMM}} = -\frac{eB}{2}\text{Tr}\{\delta_{ab}X^a X^b\} . \tag{3.7.8}$$

We now compute the angular momentum in the quantum ground state $|\psi_0\rangle$ of the CSMM. We first use the expansion of Eq. (3.4.11) to write $L_{z,\text{CSMM}}$ as

$$\begin{aligned} L_{z,\text{CSMM}} &= -\frac{eB}{2}\sum_{A=0}^{N^2-1}\delta_{ab}x_A^a x_A^b \\ &= -\hbar\delta_{ab}\Lambda^{ab} , \end{aligned} \tag{3.7.9}$$

where Λ^{ab} are the strain generators for the CSMM introduced in Sec. 3.5. We see that our derivation of the angular momentum for the CSMM theory makes sense since $\delta_{ab}\Lambda^{ab}$ is exactly the operator which generates rotations of the noncommutative coordinates X^a in the CSMM.

For the ground state of the CSMM we have $L_{z,\text{CSMM}}|\psi_0\rangle = L_0|\psi_0\rangle$ with

$$L_0 = -\hbar\left[\frac{1}{2}mN^2 + \left(\frac{1-m}{2}\right)N\right] , \tag{3.7.10}$$

and our previous results for $\langle\psi_0|\Lambda^{ab}|\psi_0\rangle$ and η_{CSMM} can be rewritten in the form

$$\langle\psi_0|\Lambda^{ab}|\psi_0\rangle = -\frac{1}{2\hbar}L_0\delta^{ab} \tag{3.7.11}$$

$$\eta_{\text{CSMM}} = \frac{1}{2}\frac{L_0}{A} . \tag{3.7.12}$$

Thus, we see that the Hall viscosity coefficient η_{CSMM} (before regularization) is equal to one half the angular momentum density $\frac{L_0}{A}$ in the ground state of the CSMM (compare with the angular momentum interpretation of the Hall viscosity from Ref. [91]). Finally, we also note that L_0 is exactly the guiding center part of the angular momentum of the Laughlin $\nu = \frac{1}{m}$ state. In the lowest Landau level the Landau orbit contribution to the angular momentum is simply $\hbar\frac{N}{2}$, which leads to the total angular momentum of the Laughlin state $L_{z,\nu=\frac{1}{m}} = \hbar\left[-\frac{1}{2}mN^2 + m\frac{N}{2}\right]$.

We now give a fluid interpretation of the superextensive (order N^2) term in L_0, which is equal to $-\frac{1}{2}\hbar mN^2$. This can be rewritten in terms of the density $\rho_0 = \frac{1}{2\pi\ell_B^2 m}$ and radius $R_N^2 \approx 2\ell_B^2 mN$ of the droplet described by the CSMM as

$$-\frac{\pi}{4}eB\rho_0 R_N^4 . \tag{3.7.13}$$

This is exactly the angular momentum of a droplet of radius R_N of the classical fluid described by the small θ limit of the NCCS action in the presence of an additional parabolic potential, as we now describe.

Recall that in the small θ limit the NCCS theory is described by the fluid action of Eq. (3.3.32). Let us add to this action a parabolic potential term which is the commutative analogue of the potential term in the CSMM action,

$$S_{para} = -\frac{eB\tilde{\omega}}{2}\rho_0 \int_0^T dt \int d^2\mathbf{x}\, \delta_{ab} X^a X^b \ , \tag{3.7.14}$$

where $\rho_0 = \frac{1}{2\pi\theta}$. The equations of motion which result from Eq. (3.3.32) plus S_{para} are $\dot{X}^1 = \tilde{\omega}X^2$ and $\dot{X}^2 = -\tilde{\omega}X^1$, as well as the constant density constraint enforced by A_0. For the initial condition $X^a(0,\mathbf{x}) = x^a$ the solution to these equations can be expressed concisely as

$$X^1(t,\mathbf{x}) + iX^2(t,\mathbf{x}) = (x^1 + ix^2)e^{-i\tilde{\omega}t} \ . \tag{3.7.15}$$

Finally, using Eq. 3.7.6 for the angular momentum we find that a droplet of radius $\mathcal{R}$ has angular momentum

$$\begin{aligned} L_{orb} &= -\int_{|\mathbf{x}|\leq\mathcal{R}} d^2\mathbf{x}\, \rho_0 \frac{eB}{2}\delta_{ab}X^a X^b \\ &= -\frac{\pi}{4}eB\rho_0\mathcal{R}^4, \end{aligned} \tag{3.7.16}$$

where “orb” stands for “orbital” since this angular momentum is associated with an overall rotation of the fluid.

We see that the superextensive term in L_0 is exactly the orbital angular momentum of a classical fluid on a commutative space in a magnetic field undergoing uniform rotational motion. Based on this observation, and using the *anisospin* $\varsigma = \frac{m-1}{2}$ defined earlier, the full angular momentum in the ground state of the CSMM can be written as

$$L_0 = L_{orb} + \hbar\varsigma N \ . \tag{3.7.17}$$

Now that we have identified the orbital contribution to the total angular momentum the remaining extensive term, which has a coefficient ς, can be interpreted as a spin angular momentum for the N particles in the fluid, in keeping with the interpretations of Hall viscosity of Refs. [87, 89, 91, 100, 101].

Now that we understand the connection between the expectation value $\langle\psi_0|\Lambda^{ab}|\psi_0\rangle$ and the total angular momentum L_0 of the state $|\psi_0\rangle$, we can give a fluid interpretation of the regularization scheme for the guiding center Hall viscosity proposed in Ref. [101]. Specifically, the regularization scheme of Ref. [101] corresponds to subtracting the orbital contribution to L_0,

$$\eta_{\text{CSMM},reg} = \frac{1}{2}\left(\frac{L_0 - L_{orb}}{A}\right) = \frac{1}{2}\hbar\varsigma\rho_0 \ . \tag{3.7.18}$$

This can be justified by noting that the classical charged fluid in a constant magnetic field and on ordinary commutative space does not exhibit a Hall viscosity[13], and so the Hall viscosity in the fluid described by the CSMM must only be due to the remaining terms in L_0 which do not have an interpretation in terms of the classical fluid on a commutative space.

3.8 Hall viscosity in the presence of anisotropy

In this section we introduce a simple modification of the CSMM which incorporates a constant unimodular metric g_{ab} (i.e., a constant metric with determinant equal to 1). This metric parametrizes an anisotropy or intrinsic geometry of a FQH state, as discussed in the works of Haldane and collaborators [89, 100, 101, 111]. As emphasized by Haldane [89, 100], introducing a unimodular metric g_{ab} into the guiding center part of a FQH state enables one to see the clear separation of the full Hall viscosity tensor η^{abcd}_{tot} into Landau orbit and guiding center contributions. When such a metric is used in the construction of the guiding center part of a FQH state, the guiding center Hall viscosity tensor η^{ab}_{H} is modified to be proportional to g^{ab} (the inverse metric of g_{ab} with $g^{ab}g_{bc} = \delta^a_c$) instead of δ^{ab}. In this section we show that for our modified CSMM, the two-index Hall viscosity tensor η^{ab}_{CSMM} is also modified to be proportional to g^{ab}. This confirms that our modification of the CSMM does indeed correspond to incorporating a nontrivial metric g_{ab} into the definition of the guiding center part of a FQH state. We also note here that the introduction of a second metric (in addition to the the metric of space) into the quantum Hall problem is exactly the starting point for the construction of the bi-metric theory of FQH states of Refs. [112, 113].

The action for our modified CSMM takes the form

$$S_{CSMM} = -\frac{eB}{2}\int_0^T dt\,\text{Tr}\Big\{\epsilon_{ab}X^a D_0 X^b + 2\theta A_0 + \tilde{\omega} g_{ab}X^a X^b\Big\} + \int_0^T \overline{\Psi}^T(i\dot{\Psi} + A_0\Psi)\,. \tag{3.8.1}$$

Note that the only change is the replacement of δ_{ab} with g_{ab} in the quadratic potential term. This is the only part of the action which could conceivably depend on a metric, since the time derivative term already uses the epsilon symbol ϵ_{ab} to contract indices. To quantize this system we make a change to a new set of variables $\tilde{X}^{\tilde{a}}$ which diagonalize the potential term but, crucially, obey the same commutation relations as the original variables. In other words, the symplectic form on the phase space of this model takes the same form in the new variables as in the old ones. Therefore the Poisson brackets and quantum commutation relations of the new variables will be identical to those for the old variables.

To describe this change of variables we decompose the metric and inverse metric in terms of coframes $e^{\tilde{a}}_a$ and

[13] This can be seen directly by writing down the equations of motion for this classical fluid in the Euler description (i.e., in terms of mass density and velocity fields), and then noting that no viscosity term is present. The Euler equations for a charged fluid in a magnetic field and a general external potential appear, for example, in Eqns. (46) and (47) of Ref. [132].

frames $E^a_{\tilde{a}}$ as

$$
\begin{aligned}
g_{ab} &= e^{\tilde{a}}_a \delta_{\tilde{a}\tilde{b}} e^{\tilde{b}}_b && (3.8.2a)\\
g^{ab} &= E^a_{\tilde{a}} \delta^{ab} E^b_{\tilde{b}} \,. && (3.8.2b)
\end{aligned}
$$

Note that we use new indices $\tilde{a}, \tilde{b} = 1, 2$ for the internal indices of the frames and coframes. The frames and coframes satisfy the relations $E^a_{\tilde{a}} e^{\tilde{a}}_b = \delta^a_b$ and $E^a_{\tilde{a}} e^{\tilde{b}}_a = \delta^{\tilde{b}}_{\tilde{a}}$, which just express the fact that the matrices e and E (with entries $e^{\tilde{a}}_a$ and $E^a_{\tilde{a}}$, respectively) are inverses of each other. In addition, it is possible to choose $\det(e) = \det(E) = 1$. This can be seen as follows. First, note that the relation between g_{ab} and $e^{\tilde{a}}_a$ can be expressed in matrix form as $g = e^T e$, where g is the matrix with entries g_{ab}. This implies that $\det(e)^2 = \det(g) = 1$, so that $\det(e) = \pm 1$. However, the parametrization of g in terms of e is invariant under the transformation $e \to Se$ for any matrix $S \in O(2)$, i.e., any S such that $S^T S = \mathbb{I}$. Then if for some reason we found a decomposition of g with $\det(e) = -1$, we can always switch to a new parametrization with $\det(e) = 1$ by replacing e with Se for any $S \in O(2)$ with $\det(S) = -1$. Then, since $E = e^{-1}$ as matrices, we also guarantee that $\det(E) = 1$.

Using the frames and coframes we introduce new matrix variables $\tilde{X}^{\tilde{a}}$ as

$$
\begin{aligned}
\tilde{X}^{\tilde{a}} &= e^{\tilde{a}}_a X^a && (3.8.3a)\\
X^a &= E^a_{\tilde{a}} \tilde{X}^{\tilde{a}} \,. && (3.8.3b)
\end{aligned}
$$

In terms of these variables we have

$$
g_{ab} X^a X^b = \delta_{\tilde{a}\tilde{b}} \tilde{X}^{\tilde{a}} \tilde{X}^{\tilde{b}} \tag{3.8.4}
$$

and, crucially,

$$
\begin{aligned}
\epsilon_{ab} X^a D_0 X^b &= \epsilon_{ab} E^a_{\tilde{a}} E^b_{\tilde{b}} \tilde{X}^{\tilde{a}} D_0 \tilde{X}^{\tilde{b}} \\
&= \det(E) \epsilon_{\tilde{a}\tilde{b}} \tilde{X}^{\tilde{a}} D_0 \tilde{X}^{\tilde{b}} \\
&= \epsilon_{\tilde{a}\tilde{b}} \tilde{X}^{\tilde{a}} D_0 \tilde{X}^{\tilde{b}} \,. && (3.8.5)
\end{aligned}
$$

We can then carry out the quantization of this modified CSMM using the $\tilde{X}^{\tilde{a}}$ variables in exactly the same way that we quantized the original CSMM in Sec. 3.4. For example we would start by expanding the $\tilde{X}^{\tilde{a}}$ in terms of a new set of real scalar variables $\tilde{x}^{\tilde{a}}_A$ ($A = 0, \dots, N^2 - 1$) exactly as in Eq. (3.4.11). This procedure results in a new ground state $|\tilde{\psi}_0\rangle$ for the modified CSMM depending on the unimodular metric g_{ab}.

We can now calculate the Hall viscosity in this modified CSMM. The setup for this calculation is the same as in

Sec. 3.5 and, in particular, we still apply an APD (or strain) to the physical position variables X^a and not the new variables $\tilde{X}^{\tilde{a}}$. The final expression for the two-index Hall viscosity tensor η^{ab}_{CSMM} is now proportional to the expectation value of the strain generators Λ^{ab} in the ground state $|\tilde{\psi}_0\rangle$ of the modified CSMM,

$$\eta^{ab}_{\text{CSMM}} = -\frac{\hbar}{A}\langle\tilde{\psi}_0|\Lambda^{ab}|\tilde{\psi}_0\rangle \ . \tag{3.8.6}$$

The expectation value $\langle\tilde{\psi}_0|\Lambda^{ab}|\tilde{\psi}_0\rangle$ is easily computed by writing $\Lambda^{ab} = E^a_{\tilde{a}}E^b_{\tilde{b}}\tilde{\Lambda}^{\tilde{a}\tilde{b}}$, where

$$\tilde{\Lambda}^{\tilde{a}\tilde{b}} = \frac{1}{4\ell_B^2}\sum_{A=0}^{N^2-1}\{\tilde{x}^{\tilde{a}}_A, \tilde{x}^{\tilde{b}}_A\} \tag{3.8.7}$$

are the strain generators for the new variables, and by noting that

$$\langle\tilde{\psi}_0|\tilde{\Lambda}^{\tilde{a}\tilde{b}}|\tilde{\psi}_0\rangle = \frac{1}{2}\left[\frac{1}{2}mN^2 + \left(\frac{1-m}{2}\right)N\right]\delta^{\tilde{a}\tilde{b}} \ , \tag{3.8.8}$$

which follows since all quantities here are in terms of the new "tilde" variables. Then the original expectation value of interest evaluates to

$$\begin{aligned} \langle\tilde{\psi}_0|\Lambda^{ab}|\tilde{\psi}_0\rangle &= \frac{1}{2}\left[\frac{1}{2}mN^2 + \left(\frac{1-m}{2}\right)N\right]\delta^{\tilde{a}\tilde{b}}E^a_{\tilde{a}}E^b_{\tilde{b}} \\ &= \frac{1}{2}\left[\frac{1}{2}mN^2 + \left(\frac{1-m}{2}\right)N\right]g^{ab} \ . \end{aligned} \tag{3.8.9}$$

After regularization, which consists of subtracting off the order N^2 term in this expectation value, the Hall viscosity tensor for the modified CSMM takes the form

$$\eta^{ab}_{\text{CSMM},reg} = -\frac{\hbar}{A}\frac{1}{2}\left(\frac{1-m}{2}\right)Ng^{ab} = \eta_{\text{CSMM},reg}\,g^{ab} \ , \tag{3.8.10}$$

where $\eta_{\text{CSMM},reg} = \frac{1}{2}\hbar\varsigma\rho_0$ as before, and where we defined $\rho_0 = \frac{N}{A}$. We find that the Hall viscosity tensor for the modified CSMM is exactly the guiding center part of the Hall viscosity tensor of the Laughlin state with nontrivial guiding center metric g_{ab} [100, 101].

We close this section by calculating the area A and the shape of the droplet of fluid described by the ground state $|\tilde{\psi}_0\rangle$ of the modified CSMM. To do this we follow the method from the end of Sec. 3.4 and consider the eigenvalue of $\text{Tr}\left\{g_{ab}X^aX^b\right\}$ when acting on the state $|\tilde{\psi}_0\rangle$. We again find that $\text{Tr}\left\{g_{ab}X^aX^b\right\}|\tilde{\psi}_0\rangle = R^2|\tilde{\psi}_0\rangle$ with the same eigenvalue R^2 from Eq. (3.4.44), and we can again interpret R^2 as a sum of contributions from N particles, $R^2 = \sum_{j=1}^N R_j^2$ with $R_j^2 = 2\ell_B^2\left(m(j-1)+\frac{1}{2}\right)$. However, the interpretation of the shape of the droplet is different now since $g_{ab}X^aX^b$ is a general quadratic form of the noncommutative position coordinates. In the simple case where $g_{ab} = \delta_{ab}$, we argued that the droplet was circular, with the j^{th} particle located somewhere on a circle of radius R_j. In this case we will argue that the droplet has the shape of an ellipse, with the particular geometry of the ellipse determined by the eigenvectors and eigenvalues of the metric g_{ab} considered as a matrix, and where the j^{th} particle is now located somewhere on an ellipse whose size is determined by R_j.

To facilitate this analysis we use a convenient parametrization [111] of the unimodular metric g_{ab} in terms of a single complex parameter $\gamma \in \mathbb{C}$ and write

$$g = \frac{1}{1-|\gamma|^2}\begin{pmatrix}(1+\gamma)(1+\overline{\gamma}) & i(\gamma-\overline{\gamma})\\ i(\gamma-\overline{\gamma}) & (1-\gamma)(1-\overline{\gamma})\end{pmatrix} . \tag{3.8.11}$$

If we also write $\gamma = \tanh(\frac{\alpha}{2})e^{i\beta}$ for real $\alpha > 0$ and a real phase β, then we find that the matrix g has the decomposition

$$g = SDS^T \tag{3.8.12}$$

with

$$S = \begin{pmatrix}\cos(\frac{\beta}{2}) & \sin(\frac{\beta}{2})\\ -\sin(\frac{\beta}{2}) & \cos(\frac{\beta}{2})\end{pmatrix} \tag{3.8.13}$$

and

$$D = \begin{pmatrix}e^{\alpha} & 0\\ 0 & e^{-\alpha}\end{pmatrix} . \tag{3.8.14}$$

Here $e^{\pm\alpha}$ are the eigenvalues of g and the columns of the matrix S are the normalized eigenvectors of g. In component form we can also write

$$g_{ab} = S_a^{\tilde{a}} D_{\tilde{a}\tilde{b}} S_b^{\tilde{b}} , \tag{3.8.15}$$

where for $S_a^{\tilde{a}}$, a indexes the rows of the matrix S and $\tilde{a}$ indexes the columns.

We now introduce new noncommutative coordinates (i.e., matrices) $Y^{\tilde{a}}$ defined as

$$Y^{\tilde{a}} = S_a^{\tilde{a}} X^a , \tag{3.8.16}$$

and in terms of these we have

$$\begin{aligned} g_{ab}X^aX^b &= D_{\tilde{a}\tilde{b}}Y^{\tilde{a}}Y^{\tilde{b}} \\ &= e^{\alpha}(Y^1)^2 + e^{-\alpha}(Y^2)^2 . \end{aligned} \tag{3.8.17}$$

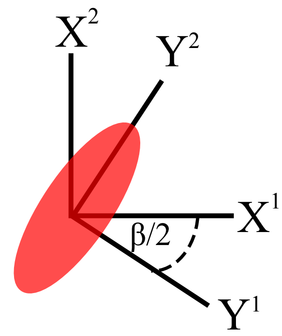

Figure 3.1: The shape and orientation of the droplet of fluid which is described by the ground state $|\tilde{\psi}_0\rangle$ of the modified CSMM incorporating the unimodular spatial metric g_{ab}.

We now see that in the modified CSMM with metric g_{ab}, we can interpret the j^{th} particle as residing on an ellipse with the lengths of the minor and major axes of that ellipse given by $r_{1,j} = e^{-\frac{\alpha}{2}}R_j$ and $r_{2,j} = e^{\frac{\alpha}{2}}R_j$[14]. Furthermore, this ellipse has its minor and major axes lined up with the axes of the $Y^{\tilde{a}}$ coordinate system, which is rotated from the X^a coordinate system by an angle of $\frac{\beta}{2}$ as shown in Fig. 3.1. The area of the ellipse where the j^{th} particle is located is $\pi r_{1,j}r_{2,j} = \pi R_j^2$, and since R_j^2 is linear in j, we again find that the particle density is constant inside the droplet. Finally, the area of the droplet is equal to the area of the ellipse for particle N which is $A = \pi R_N^2 \approx 2\pi\ell_B^2 mN$, just as in the ordinary CSMM.

We conclude that the modified CSMM incorporating the unimodular metric g_{ab} describes an elliptical droplet of fluid with the same area A and constant density ρ_0 as the ordinary CSMM, and where the details of the shape of the ellipse are determined by the eigenvalues and eigenvectors of the metric g_{ab}. In addition, since the density ρ_0 is the same as for the original CSMM, we find that the coefficient $\eta_{\text{CSMM},reg} = \frac{1}{2}\hbar\varsigma\rho_0$ of Hall viscosity for the CSMM with $g_{ab} \neq \delta_{ab}$ is numerically equal to the coefficient for the case where $g_{ab} = \delta_{ab}$. The only difference between these two cases is the structure of the Hall viscosity tensor, since for $g_{ab} \neq \delta_{ab}$ the two index tensor $\eta^{ab}_{\text{CSMM},reg}$ is proportional to g^{ab} instead of δ^{ab}.

3.9 Conclusion

In this Chapter we investigated the geometric properties of the Laughlin FQH states within the CSMM description of these states which, roughly speaking, models these states as a charged fluid in a magnetic field and propagating on a noncommutative space. We focused our attention on the specific properties of Hall viscosity, Hall conductance in a

[14]Recall that the equation $a^2x^2 + b^2y^2 = R^2$ describes an ellipse in the (x, y) plane with the lengths of the two axes of the ellipse given by $\frac{R}{a}$ and $\frac{R}{b}$.

non-uniform electric field, and the Hall viscosity in the presence of anisotropy. We found that the answers for these quantities calculated from the CSMM description contain only the *guiding center* contribution to the known answers for these quantities in the Laughlin states.

These results lead us to the general conclusion that the CSMM description of the Laughlin FQH states accurately captures the guiding center contribution to the geometric properties of these states, but lacks the Landau orbit contribution. As we remarked in the Introduction, the Landau orbit contribution is often considered to be a trivial contribution since the interesting correlations in the Laughlin state are contained in the guiding center part of its wave function/state vector. Therefore we find that the CSMM description captures the most important contribution, namely the guiding center contribution, to the physics of the Laughlin FQH states. However, any attempt to completely describe the Laughlin states using the CSMM or NCCS theory must also include some auxiliary degrees of freedom which account for the missing Landau orbit contributions to the geometric properties of these states.

There are several possible directions for future work in this area. One direction would be to continue to develop the fluid interpretation of the CSMM. One goal of this work would be to find an appropriate definition of a density operator $\rho(\mathbf{x})$ which is a function of a commutative two-dimensional coordinate $\mathbf{x} \in \mathbb{R}^2$ and which is defined on length scales much larger than the scale set by θ in the noncommutative theory. One could then check whether this density operator satisfies the Girvin-Macdonald-Plaztman algebra, and also attempt to compute the static structure factor and compare to the known answer for the Laughlin states [133]. Another goal of this work would be to connect the CSMM description of the Laughlin states with a different fluid description of these states, which is Wiegmann's vortex fluid description [134]. In this description the Laughlin FQH state with N electrons is modeled as a rotating incompressible fluid containing N point vortices each carrying a quantized circulation Γ which depends on the filling fraction of the Laughlin state. On this topic we note that Bettelheim has recently introduced a method for defining density and velocity fields in the CSMM which are functions of a commutative coordinate $\mathbf{x}$ in Ref. [135], and it would be interesting to develop his approach further and to use it to connect with Wiegmann's vortex fluid description. We also note that the problem of defining density operators in NCCS theory and the CSMM has been considered before in Refs. [120, 122].

A second direction for future work would be to investigate the Hall viscosity and other geometric response properties in matrix models which describe other more complicated FQH states. For example, a matrix model for the Jain states [136] has been proposed in Ref. [137]. More recently, the authors of Ref. [131] proposed a class of matrix models for the *Blok-Wen* series of non-Abelian FQH states [138]. It would also be interesting to search for new matrix models which can describe other FQH states of interest.

Chapter 4

Topological electromagnetic responses of bosonic quantum Hall, topological insulator, and chiral semi-metal phases in all dimensions[1]

4.1 Introduction

In the years since the theoretical prediction and experimental discovery of the electron topological insulators[139, 140], the study of symmetry-protected topological (SPT) phases of matter [10–12, 14, 15] has emerged as an extremely rich subfield of condensed matter physics, with interesting and surprising connections to high-energy physics and mathematics. Although there has been tremendous progress in the understanding of these states of matter, some basic issues about these phases are still the subject of intense investigation. As illustrative examples we point to the question of which theories can describe a surface termination of the time-reversal invariant electron topological insulator in three spatial dimensions[141–147], as well as the analogous question for the surface of the bosonic topological insulator in three spatial dimensions[34, 148].

A very useful definition of an SPT phase is as follows[13]. Consider a quantum many-body system with Hamiltonian H, where H has the symmetries of a group G and a gapped spectrum. Then the ground state $|\Psi\rangle$ of H represents an SPT phase if it satisfies several properties. First, $|\Psi\rangle$ should be unique independently of the topology of the (closed) spatial manifold that H is defined on. This ensures that the ground state of H does not represent a phase with topological order (no excitations with fractional charge or statistics, etc.). Second, $|\Psi\rangle$ should be invariant under the action of G, i.e., $U(g)|\Psi\rangle = |\Psi\rangle$ for any $g \in G$, where $U(g)$ is a representation of G on the Hilbert space of the system. This means that the ground state of H does not spontaneously break the symmetry of the group G. Finally, $|\Psi\rangle$ cannot be continuously tuned to a *trivial* product state (e.g., by adding terms to the Hamiltonian) without (i) breaking the symmetry of G, or (ii) closing the gap in the spectrum of H. Despite the lack of anyon excitations in the bulk, interesting degrees of freedom will in general be present at the boundary of an SPT phase.

In this Chapter we focus our primary attention on bosonic SPT phases and, in particular, on those bosonic SPT

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phases which are analogues of more familiar topological phases of fermions. We are especially interested in the Bosonic Integer Quantum Hall (BIQH) effect[9, 18, 149–157], a bosonic analog of the ordinary $\nu = 1$ Integer Quantum Hall effect of fermions in three (spacetime) dimensions, and in the time-reversal invariant Bosonic Topological Insulator (BTI)[19, 20, 158, 159], a bosonic analogue of the time-reversal invariant electron topological insulator in four dimensions. In fact, the main goal of this Chapter is to consider generalizations of the BIQH and BTI states to *all* odd and even spacetime dimensions, respectively, and then to study the physical properties of these higher-dimensional states. The reader should note that in the remainder of this Chapter the word "dimension" always refers to the *spacetime* dimension. We always write "spatial dimension" when we want to discuss the dimension of space only.

BIQH phases are protected only by $U(1)$ charge-conservation symmetry, while the BTI phase is protected by the symmetry group $U(1) \rtimes \mathbb{Z}_2$, where, as we discuss later, the $\mathbb{Z}_2$ symmetry is unitary charge-conjugation symmetry $\mathbb{Z}_2^C$ in dimensions $2, 6, 10$, etc., and anti-unitary time-reversal symmetry $\mathbb{Z}_2^T$ in dimensions $4, 8, 12$, etc. The symbol "$\rtimes$" means that the $U(1)$ and $\mathbb{Z}_2$ symmetry operations do not commute with each other. Since both of these phases have $U(1)$ charge-conservation symmetry, they can both be coupled to an external electromagnetic field A_μ. One can then study the electromagnetic response of these states.

One of our main results in this Chapter is an explicit derivation of the (topological part of) the electromagnetic response of BIQH phases in all odd dimensions and BTI phases in all even dimensions. From a physical standpoint the magnitude of the electromagnetic response is extremely interesting, as it is known already in three dimensions that the requirement that a BIQH state have no topological order places a constraint on the allowed values of the Hall conductance of any putative BIQH state[18]. In particular, the Hall conductance must be a multiple of 2 (in units of $\frac{e^2}{h}$), i.e., a BIQH state has twice the Hall conductance that a free fermion Integer Quantum Hall state can have. In higher dimensions we also find that the electromagnetic response of the BIQH state is some integer multiple of the minimum value which can be realized by free fermions, and we find analogous results in even dimensions for BTI states.

To calculate the electromagnetic response of these states, we need a concrete model to work with. For reasons to be discussed in the next section, we choose to use the Nonlinear Sigma Model (NLSM) description of bosonic SPT phases[35, 36, 160–163]. This allows us to use the theory of gauged Wess-Zumino (WZ) actions[37–41, 164] to study the boundary of these states, and from our study of the boundary we are able to deduce the bulk response. As a byproduct, our explicit construction of gauged WZ actions for the boundaries of these states allows us to study several physical properties of these states in more detail. We show that the boundary theory for the BTI displays a bosonic analogue of the parity anomaly for Dirac fermions in odd dimensions[58–60, 70, 165], and we also use the boundary theory of the BIQH state to construct effective theories for bosonic analogues of Weyl (or chiral) semi-metals in all even dimensions.

For the case of the BIQH state, we also provide an alternative derivation of the response by requiring the gauge invariance of (the exponential of) the Chern-Simons functional describing the electromagnetic response of the state. We also use this gauge invariance argument to derive and discuss the electromagnetic and gravitational responses of Fermionic Integer Quantum Hall (FIQH) phases in different dimensions. This gauge invariance argument provides us with a general understanding of the difference in the quantization of response coefficients of BIQH and FIQH phases.

Before moving on, we take this opportunity to provide some justification for our study of bosonic SPT phases in dimensions higher than the physically relevant dimensions of two, three, and four. Studying a state of matter in generic dimensions can often reveal underlying organizational principles or mathematical structures which cannot be seen by studying low-dimensional examples on their own. An obvious example of this is the periodic table of topological insulators and superconductors[166, 167], which exhibits an eightfold periodicity in the dimension of space (i.e., the pattern does not completely develop if one considers only low dimensions). In the case of bosonic SPT phases, low-dimensional examples suggest that the response of the bosonic analogue of a given fermionic state (Integer Quantum Hall or electron topological insulator) is twice that of its fermionic counterpart. However, our results in this Chapter clearly show that this is *not* the case in higher dimensions. Finally, it is also worth mentioning that many new insights on four-dimensional physics can be gained by imagining that our four-dimensional spacetime is the boundary of a five-dimensional SPT phase[168–170].

This Chapter is organized as follows. First, in Sec. 4.2 we outline our basic approach and summarize our main results. In Sec. 4.3 we review the relevant background information on BIQH and BTI phases, the NLSM description of SPT phases, and the method of gauged WZ actions. In Sec. 4.4 we construct the gauged WZ action for the boundary of the BIQH phase, and we use the anomaly of the gauged boundary action to deduce the bulk response of the BIQH phase. We also give an alternative derivation of the BIQH response which relies on only the bulk physics of the NLSM. In Sec. 4.5 we use a general gauge invariance argument to understand the electromagnetic response of BIQH states, and also the electromagnetic and gravitational responses of FIQH states in odd dimensions. In particular, we illuminate the important differences between the quantization of response coefficients in BIQH and FIQH phases. In Sec. 4.6 we construct the gauged WZ action for the boundary of the BTI phase, and we use the gauged boundary action to study the symmetry-breaking BIQH response of the BTI boundary. In Sec. 4.7 we use the results from Sec. 4.4 and Sec. 4.6 to (i) construct effective theories for bosonic analogues of Weyl, or chiral, semi-metals in all even dimensions, (ii) show that the boundary of a BTI state displays an analogue of the parity anomaly for Dirac fermions in odd dimensions, and (iii) study the physics of symmetry-breaking domain walls on the boundary of BTI states. Sec. 4.8 presents our conclusions. Finally, in a series of Appendices we examine the results of the Chapter from a more mathematical point of view, and also derive several important formulas which are used throughout the

Chapter.

4.2 Basic approach and Summary of Results

In this section we outline our basic approach to calculating the electromagnetic response of higher-dimensional bosonic SPT phases, and then we present our results. In this Chapter we work in units where $\hbar = e = 1$, where e is the charge of the basic particles (bosons or fermions) which make up the state we are interested in. To restore e in any formula one can simply replace A_μ (the external electromagnetic field) with eA_μ.

Let us first discuss the general form that the topological part of the electromagnetic response is expected to take for BIQH and BTI states. In odd dimensions, the response of a higher-dimensional analogue of a Quantum Hall state to an external field $A = A_\mu dx^\mu$ (we use differential form notation) is characterized by a Chern-Simons (CS) term $S_{CS}[A]$ in the effective action for the external field. In $2m-1$ dimensions this term takes the form

$$S_{CS}[A] = \frac{N_{2m-1}}{(2\pi)^{m-1} m!} \int_{\mathcal{M}} A \wedge F^{m-1} \ , \tag{4.2.1}$$

where N_{2m-1} is called the *level* of the CS term, $F = dA$, F^{m-1} is shorthand for the wedge product of F with itself $m-1$ times, and $\mathcal{M}$ represents the spacetime manifold. Let us also note here that all actions in the Chapter are written down in Minkowski signature (real time) except in Sec. 4.5 and Appendix C.2, where we consider CS and other terms in Euclidean spacetimes. On the other hand, the response of an analogue of a topological insulator in $2m$ dimensions is characterized by a "Chern character" term (we avoid using "theta-term" here since that name is also used for a type of topological term in the NLSM action),

$$S_{CC}[A] = \frac{\Theta_{2m}}{(2\pi)^m m!} \int_{\mathcal{M}} F^m \ . \tag{4.2.2}$$

Here the coefficient Θ_{2m} should be interpreted as an angular variable, although its period is not necessarily 2π. We call this term a "Chern character" term as the quantity $\frac{1}{m!}\left(\frac{F}{2\pi}\right)^m$ appears as the m^{th} term in the expansion of the total Chern character $\text{ch}[F] = e^{\frac{F}{2\pi}}$ of a $U(1)$ principal bundle with curvature F. [56] Since locally we can write $F^m = d(A \wedge F^{m-1})$, we see that for a BTI state with a boundary, the term $S_{CC}[A]$ can be interpreted as a CS term at level $\frac{\Theta_{2m}}{2\pi}$ on the $(2m-1)$-dimensional boundary of the BTI state (more precisely, this is only true when the bulk field configuration F has vanishing topological contributions).

For the analogues of Integer Quantum Hall states of fermions (FIQH states) in odd dimensions, the level N_{2m-1} of the CS term can be any integer[51–54], while for free fermion topological insulators, and their generalizations to higher dimensions, the angle Θ_{2m} is 2π-periodic and the value which represents a non-trivial topological insulator state is $\Theta_{2m} = \pi$[8] (the result for fermionic topological insulators in any even dimension is easily established using the axial anomaly for a Dirac fermion in $2m$ dimensions). For bosonic SPT phases in low dimensions we know that $N_3 = 2k$, $k \in \mathbb{Z}$ for BIQH states in three dimensions[9, 18], that Θ_4 has 4π-periodicity, and $\Theta_4 = 2\pi$ for the non-trivial BTI state in four dimensions[19, 20, 171, 172].

One of the main purposes of this Chapter is to calculate the values of the response coefficients N_{2m-1} and Θ_{2m} for BIQH and BTI states in all dimensions. There are (at least) two ways that one could go about doing this. The first way would be to formulate a general physical argument based, for example, on the consistency of the value of N_{2m-1} or Θ_{2m} and the fact that a bosonic SPT state should have no fractionalized excitations, and in this way determine a constraint on the possible values of N_{2m-1} or Θ_{2m}. In fact, such an argument has already been given for the BIQH state in the case $m = 2$ (three spacetime dimensions). In Ref. [18] the authors showed that if the response coefficient N_3 (which is just the Hall conductance in units of $\frac{e^2}{h}$) is odd, then the underlying theory must contain an excitation of charge one (in units of the charge e of the underlying bosons) with fermionic exchange statistics. An excitation with fermionic statistics is not allowed in a state of bosons which has no fractionalized excitations, and so the authors of Ref. [18] concluded that N_3 must be an even integer for BIQH states in three dimensions. Generalizing this argument to higher dimensions clearly represents a significant conceptual difficulty, as in higher dimensions one is probably forced to consider generalized braiding processes for extended objects such as string or membrane excitations[173–175]. For this reason we do not pursue this approach in this work, and instead use a second method.

The second method for answering this question, and the method that we choose to use, is to (i) start with a concrete field-theoretic model which is believed to accurately describe the low-energy physics of a BIQH or BTI state in the relevant dimension, (ii) couple this model to the external field A, and (iii) directly calculate the electromagnetic response for this particular model. In the literature there are two main kinds of field-theoretic models that can describe SPT phases: topological quantum field theory (TQFT) in terms of gauge field variables (e.g., Chern-Simons theory in three dimensions[9, 154, 176, 177] and twisted gauge theory[68, 178, 179] in four dimensions[159, 180]) and the Nonlinear Sigma Model (NLSM) description in terms of constrained bosonic fields [35, 36, 160–163]. In both approaches the bulk topological order is trivial but global symmetry is imposed nontrivially on the field variables. In this Chapter we choose to use the NLSM description since this description can be easily generalized to any spacetime dimension.

In the NLSM description, a bulk bosonic SPT phase in $d + 1$ spacetime dimensions is described by an $O(d + 2)$ NLSM with topological theta term having coefficient $\theta = 2\pi k$ where $k \in \mathbb{Z}$. In this description the boundary of the SPT phase is then described by an $O(d + 2)$ NLSM with Wess-Zumino (WZ) term, where the coefficient of the WZ term, known as the *level* of the WZ term, is equal to k. Conventionally, writing down the WZ term in the boundary theory requires defining an extension of the NLSM field into an auxiliary direction of spacetime. In a series of works[35, 36, 160–163], the NLSM description has been shown to accurately describe the structure of the ground state wave function of SPT phases[13], the point and loop braiding statistics of excitations in gauged SPT phases[13, 173, 174, 181, 182], the decorated domain wall construction of SPT phases[183], as well as several other properties of these phases. In addition, a mathematical classification of bosonic SPT phases based on the NLSM description has been shown to be completely identical to the group cohomology classification[14] in situations where both classification schemes can be applied. In fact, there is even a concrete procedure for calculating the cocycle which classifies an SPT phase in the group cohomology approach by starting with the NLSM description of that SPT phase[184]. Additional applications of NLSMs to the study of SPT phases with translation symmetry and to exotic quantum phase transitions in Weyl semi-metals were considered recently in Refs. [185, 186]. However, despite the many successes of the NLSM description, deriving the electromagnetic response of a bosonic SPT phase *directly* from its NLSM description remains a difficult problem. In the few instances in which the response of an SPT phase has been determined from its NLSM description it has been by an indirect method such as an appeal to gauge invariance of the final effective action[187], a dual vortex description of the theory[19], or a description of the NLSM involving auxiliary fermions which also carry charge of the external field A[34, 188]. The descriptions in terms of auxiliary fermions are in turn based on a set of formulas due to Abanov and Wiegmann[189] which allow one to generate an $O(d+2)$ NLSM with theta term by coupling the NLSM field to a set of auxiliary fermions and then integrating out those fermions.

In this Chapter we overcome this difficulty and give a direct computation of the response of higher-dimensional generalizations of BIQH and BTI states in all dimensions from their NLSM description. To do this we use a two-pronged approach. First, instead of focusing on the bulk of the SPT phase, we study the boundary, and in particular, the behavior of the gauged boundary theory. In the case of the BIQH state we find that the boundary has a perturbative $U(1)$ anomaly, which we explicitly calculate. Since the CS action changes by a boundary term under a gauge transformation, requiring the entire system (bulk plus boundary) to be gauge-invariant allows us to determine the bulk response coefficient N_{2m-1} from the boundary anomaly. In the BTI case we show that the boundary exhibits a Quantum Hall response when the associated discrete symmetry (e.g., time-reversal in four dimensions) of the BTI state is broken. Again, from this boundary response we can directly read off the coefficient Θ_{2m} using the fact that for a system with boundary, the action $S_{CC}[A]$ is equivalent to a CS action with level $\frac{\Theta_{2m}}{2\pi}$ on the boundary of the BTI.

To study the boundary theory coupled to the external field electromagnetic A we use the method of gauged WZ actions [37–41] (see also Refs. [190, 191] for some recent applications of gauged WZ actions in condensed matter physics). This machinery can be applied to this problem since, in the NLSM description, the boundary of an SPT phase in $d+1$ dimensions is described by an $O(d+2)$ NLSM with WZ term. Therefore we require knowledge of the proper way to gauge a WZ action in order to gauge the boundary theory of the SPT phase. For readers who are familiar with gauged WZ actions it is also worth remarking that all terms in the gauged actions we write down (with the sole exception of the original un-gauged WZ term) are expressed as integrals of fields only over the physical boundary spacetime. That is, we do not assume an extension of the external field A into the auxiliary direction of spacetime which is used to write down the WZ term. This is to be contrasted with the general approach of Ref. [41], in which all terms in the gauged action are written as integrals over the extended spacetime, and an analogue of the method used to obtain the Chern-Simons form from the Chern character must then be used to reduce the terms in the action to integrals only over the physical spacetime. This difficulty usually prevents one from writing down an explicit local (i.e., not involving integrals over the extended spacetime) form for the gauged WZ action in any dimension. We emphasize that here we do not encounter this difficulty. For the BIQH and BTI systems that we study, we give explicit local expressions for the gauged boundary action in all dimensions.

In Sec. 4.4 we use this method to derive the unusual result that for BIQH states in $2m-1$ dimensions the level of the CS term in the effective action for A is quantized as

$$N_{2m-1} = (m!)k \ , \ k \in \mathbb{Z} \ , \tag{4.2.3}$$

where $m!$ denotes the factorial of m. This general formula agrees with existing results for the cases of three[9, 18, 154] and five[188] dimensions ($m = 2$ and $m = 3$, respectively), and gives a prediction for all higher odd dimensions. In this case we also provide an alternative derivation of the value of N_{2m-1} using only the NLSM description of the *bulk* of the BIQH state, which confirms our result derived using the anomaly of the boundary theory.

Next, in Sec. 4.5 we show that the BIQH response computed in Sec. 4.4 can be understood by requiring the exponential of the CS response action for the BIQH state to be invariant under large $U(1)$ gauge transformations when the response theory is formulated on general closed, compact Euclidean manifolds. Furthermore, we apply these gauge invariance arguments to study the electromagnetic *and* gravitational responses of fermionic SPT phases with $U(1)$ symmetry in odd dimensions, and point out the distinctive features between the bosonic and fermionic cases.

Moving on to the BTI case, we show in Sec. 4.6, using the NLSM description of the BTI phase, that the non-trivial BTI state in $2m$ dimensions is characterized by a coefficient

$$\Theta_{2m} = 2\pi \left(\frac{m!}{2} \right) \ . \tag{4.2.4}$$

Again, this general formula agrees with the known answer in four dimensions[19, 20, 171, 172] ($m = 2$) and gives a prediction for all higher even dimensions. It also suggests that the period of the parameter Θ_{2m} is $2\pi(m!)$ for BTI states in $2m$ dimensions.

In Sec. 4.7 we use the gauged boundary actions derived in Sec. 4.4 and Sec. 4.6 to derive several other interesting results. First, we construct an effective theory for a bosonic analogue of a two-node Weyl (or chiral) semi-metal in all even dimensions d using two copies of the boundary action for the BIQH state. We refer to this state as a bosonic chiral semi-metal (BCSM). The theory that we construct has an electromagnetic response of the form ($\mathbb{R}^{d-1,1}$ is d-dimensional Minkowski spacetime)

$$S^{(b)}_{eff}[A,B] = -2\left(\frac{d}{2}+1\right)\frac{1}{(2\pi)^{\frac{d}{2}}}\int_{\mathbb{R}^{d-1,1}} B \wedge A \wedge (dA)^{\frac{d}{2}-1} \tag{4.2.5}$$

where $B = B_\mu dx^\mu$ is a one-form whose components B_μ represent the separation in energy and momentum of the two copies of the BIQH boundary theory (in the fermionic case the components of B_μ specify the separation in energy and momentum of the two Weyl cones). This response is larger than the response of the fermionic chiral semi-metal in the same dimension by a factor of $\left(\frac{d}{2}+1\right)!$. This factor turns out to be identical to the factor of $m!$ discussed earlier for the BIQH state, since our semi-metal theory in d dimensions is constructed from two copies of the boundary theory for the BIQH state in $d+1 = 2m-1$ dimensions. Next, we show that the boundary theory of the BTI exhibits a bosonic analogue of the parity anomaly of a single Dirac fermion in odd dimensions[58–60, 70, 165]. This parity anomaly is essentially the statement that although the boundary theory of the BTI is gauge-invariant and possesses the $\mathbb{Z}_2$ symmetry of the BTI state, the $\mathbb{Z}_2$ symmetry can be *spontaneously* broken at the boundary of the BTI, resulting in a half-quantized BIQH response on the boundary. This anomaly then provides strong evidence that the boundary theory of the BTI (with the symmetries of the BTI phase) cannot be realized intrinsically in $2m-1$ dimensions. Finally, we analyze the physics of symmetry-breaking domain walls on the boundary of the BTI state, and we show that the physics of such domain walls provides a nice example of the phenomenon of *anomaly inflow*[75] in bosonic SPT phases.

The Appendices of the Chapter contain several additional results, most of a more mathematical nature. In Appendix C.1 we use the well-known connection between gauged WZ actions and equivariant cohomology to understand the mathematical structure of the gauged WZ actions that we construct for the boundaries of BIQH and BTI states. In particular, we show that the construction of these actions is related to the mathematical problem of constructing an *equivariant extension* of the volume form for the sphere S^{2m-1} (in the BIQH case) or S^{2m} in the BTI case, and we study this mathematical problem in detail. In Appendix C.2 we show an example of the computation of the Chern character for the field strength F on the complex projective space $C\mathbb{P}^m$. This example serves to illustrate the necessity of the peculiar quantization of the CS level required for gauge invariance of the CS term on general manifolds as derived in Sec. 4.5. In Appendix C.3 we discuss a dimensional reduction procedure which allows one to obtain the response action for the BTI phase from the response action for the BIQH phase in one higher dimension. In Appendix C.4 we derive a general dimensional reduction formula for topological theta terms in NLSMs. Finally, in Appendix C.5 we compute the electromagnetic response of the $O(2)$ NLSM in one dimension.

4.3 Background

In this section we introduce the relevant background material necessary for understanding the later sections of the Chapter. We start with a brief review of the physics of the BIQH and BTI states, and also present definitions of higher-dimensional generalizations of these states. We then review the NLSM description of the bulk and boundary of bosonic SPT phases, and discuss the specifics of the NLSM descriptions of the BIQH and BTI states that we study in this Chapter. Finally, we give a general discussion of the tool of gauged WZ actions, and we describe in concrete terms the procedure that we use in this Chapter to construct gauged WZ actions for the boundaries of BIQH and BTI states.

4.3.1 BIQH and BTI phases

In its original formulation[9, 18], the BIQH phase was conceived of as a gapped quantum phase of bosons in three spacetime dimensions which exhibits a non-zero Hall conductance, but does not have any bulk topological order. As an SPT phase it is protected by only charge-conservation symmetry, i.e., we have $G = U(1)$ where G is the symmetry group of the SPT phase. Physically, the BIQH state is characterized by a CS term in the effective action for the external field A,

$$S_{eff}[A] = \frac{N_3}{4\pi}\int_{\mathcal{M}} A \wedge dA \ , \tag{4.3.1}$$

in which the coefficient N_3 (which is just the Hall conductance in units of $\frac{e^2}{h}$) is quantized in integer multiples of 2. The authors of Ref. [18] gave a very appealing physical argument for why the value of $N_3 = 1$ is not allowed if the BIQH state is required to have no fractionalized excitations, and we now briefly review their argument. Consider a hypothetical BIQH state on flat space, and a configuration of A in which a thin tube of 2π flux pierces the spatial surface. According to the action $S_{eff}[A]$, the point in space where the flux tube pierces the plane will bind a charge equal to N_3. Now one invokes a standard argument[2] that 2π flux is gauge-equivalent to zero flux, and so the point-like excitation created by threading the flux is in fact an excitation of the BIQH fluid and not an external defect. One can therefore ask about the phase obtained by the wavefunction of the system after a process in which two such excitations are exchanged. By the Aharanov-Bohm effect, taking one excitation completely around another results in a statistical phase of $2\pi N_3$. The phase for an exchange process is therefore half of that, $\vartheta_{ex} = \pi N_3$. From this result the authors

[2] In fact, this statement is only true on a lattice when we can couple to a compact $U(1)$ gauge field, or in the continuum when the level of the CS term is an integer. To see what can go wrong, consider $N_3 = \frac{1}{q}$ for $q \in \mathbb{Z}$. Then the object created by threading a thin 2π flux tube has charge $\frac{1}{q}$ and so only q such fluxes are a physical excitation of the system, in the sense that all states in the physical Hilbert space of the quantum mechanical system should have integer charge.

of Ref. [18] concluded that the state described by the effective action of Eq. (4.3.1) contains a fermionic excitation if N_3 is odd, and so N_3 must be an even integer in order for the action of Eq. (4.3.1) to represent the electromagnetic response of a BIQH phase.

In this Chapter we consider generalizations of the BIQH state to all odd spacetime dimensions. One definition of a BIQH state in $2m-1$ dimensions which is sufficient for our purposes is that a BIQH state is an SPT phase of bosons which is protected by the symmetry group $G = U(1)$, where $U(1)$ is charge conservation symmetry, and which exhibits a CS response to an applied electromagnetic field A of the form of Eq. (4.2.1). We should also mention here that in odd dimensions there is a countable infinity of different BIQH states, i.e., these states have a $\mathbb{Z}$ classification[14, 35]. This means that the coefficient N_{2m-1} takes on a countable infinity of values which all have the form of some particular number times an integer.

On the other hand, the BTI phase[14, 19, 159] is a bosonic analogue of the time-reversal invariant electron topological insulator in four spacetime dimensions. As an SPT phase it is protected by the symmetry group $G = U(1) \rtimes \mathbb{Z}_2^T$, where $U(1)$ represents charge conservation and $\mathbb{Z}_2^T$ is time-reversal symmetry. If we write $\mathbb{Z}_2^T = (1, \mathcal{T})$ where $\mathcal{T}$ is the time-reversal operator, then we have $\mathcal{T}^2 = 1$ for the BTI. This should be contrasted with the relation $\mathcal{T}^2 = (-1)^F$ which holds for the electron topological insulator, where F is the fermion number. The semi-direct product "$\rtimes$" indicates that the $U(1)$ and $\mathbb{Z}_2^T$ symmetries do not commute with each other. In the next subsection we will see an explicit representation of the action of the group $U(1) \rtimes \mathbb{Z}_2^T$ on the fields in the NLSM description of the BTI.

The bulk of the BTI phase is characterized by an effective action for A of the Chern character form

$$S_{eff}[A] = \frac{\Theta_4}{8\pi^2} \int_{\mathcal{M}} F \wedge F \ , \tag{4.3.2}$$

where $F = dA$, and $\Theta_4 = 2\pi$ for the BTI (compare with $\Theta_4 = \pi$ for the electron topological insulator[8]). The parameter Θ_4 has 2π-periodicity in the case of the electron topological insulator[8] but 4π-periodicity in the BTI case[19, 20]. One way to understand this effective action is to consider what happens when the spacetime $\mathcal{M}$ has a boundary $\partial\mathcal{M}$. In this case, if the bulk field configuration F is topologically trivial, then we can write $F \wedge F = d(A \wedge dA)$ to find

$$S_{eff}[A] = \frac{\Theta_4}{2\pi} \frac{1}{4\pi} \int_{\partial\mathcal{M}} A \wedge dA \ , \tag{4.3.3}$$

which is equivalent to a Quantum Hall state with Hall conductance $\sigma_H = \frac{\Theta_4}{2\pi}$ on the boundary of $\mathcal{M}$. In particular, for the BTI we have $\Theta_4 = 2\pi$ so that the surface of the BTI exhibits a *half-quantized* BIQH effect (i.e., $\sigma_H = 1$ on the surface). Such a surface Quantum Hall response breaks the time-reversal symmetry of the BTI.

Now we turn to the question of how to generalize the BTI state to all even dimensions. The main issue with generalizing the BTI state to all even dimensions is that the discrete part of the symmetry group G, which was anti-unitary time-reversal symmetry $\mathbb{Z}_2^T$ in four dimensions, should be chosen differently when the spacetime dimension is equal to zero or two modulo four. Whenever the spacetime dimension is equal to zero modulo four we choose the discrete part of G to be anti-unitary time-reversal symmetry $\mathbb{Z}_2^T$. On the other hand, whenever the spacetime dimension is equal to two modulo four we choose the discrete part of G to be *unitary* charge-conjugation (or particle-hole) symmetry $\mathbb{Z}_2^C$. This choice is consistent with the results of the group cohomology[14] and NLSM[35] classifications of SPT phases in these dimensions, and with the symmetries which protect the fermion topological insulators in two and four spacetime dimensions, respectively[8].

We therefore choose to use the following definition of a BTI phase in all even dimensions. A BTI phase in spacetime dimension $2m$ is an SPT phase of bosons with symmetry group

$$G = \begin{cases} U(1) \rtimes \mathbb{Z}_2^T & , \ m = \text{even} \\ U(1) \rtimes \mathbb{Z}_2^C & , \ m = \text{odd} \end{cases} , \tag{4.3.4}$$

and which exhibits a bulk response to an external field A of the form of Eq. (4.2.2). As we noted earlier, when the spacetime $\mathcal{M}$ has a boundary $\partial\mathcal{M}$, and when the field configuration F is topologically trivial, this bulk response is equivalent to a boundary Quantum Hall response of the form of Eq. (4.2.1) with coefficient $N_{2m-1} = \frac{\Theta_{2m}}{2\pi}$. In addition, this boundary Quantum Hall response breaks the $\mathbb{Z}_2^T$ symmetry (for m even) or $\mathbb{Z}_2^C$ symmetry (for m odd) of the BTI phase. When we discuss the BTI phase in a general dimension $2m$, and when we do not have a particular m in mind, we just write $\mathbb{Z}_2$ for the discrete part of G. However, the reader should always keep in mind that the $\mathbb{Z}_2$ symmetry is different for the cases of m even and m odd as discussed in this subsection.

Finally, we also mention that based on the group cohomology[14] and NLSM[35] classification schemes, only the smallest value of Θ_{2m} is expected to represent a non-trivial BTI phase in $2m$ dimensions. This can be understood as follows. For SPT phases with $U(1) \rtimes \mathbb{Z}_2^T$ symmetry in four dimensions the group cohomology and NLSM classifications predict a $(\mathbb{Z}_2)^2$ classification. One of these $\mathbb{Z}_2$ factors corresponds to the BTI state, while the other corresponds to a state in which the $U(1)$ symmetry plays no role[20] (so this second state cannot be interpreted as an insulator). This means that there is only a single non-trivial BTI state in four dimensions. In addition, in two dimensions the classification for SPTs with $U(1) \rtimes \mathbb{Z}_2^C$ symmetry is $\mathbb{Z}_2$, and the $U(1)$ symmetry does play a role in the non-trivial phase, so we identify that phase with the BTI phase in two dimensions. Based on this evidence we expect the existence of a single non-trivial BTI phase to generalize to all even dimensions. In the context of the NLSM classification this can be understood as coming from the fact that in $2m$ dimensions the $O(2m+1)$ NLSM theory with $\theta = 2\pi k$ can be smoothly connected to the theory with $\theta = 2\pi(k \pm 2)$ (see, e.g., the discussion in Ref. [35]).

4.3.2 NLSM description of the bulk and boundary of SPT phases

We now give a brief review of the NLSM description of SPT states, which was presented in its fully developed form in Ref. [35]. Let us consider bosonic SPT phases in $d+1$ spacetime dimensions. The spacetime coordinates are x^μ, $\mu = 0,\dots,d$ ($x^0 = t$ is the time coordinate), and for now we focus on the case of flat Minkowski spacetime $\mathbb{R}^{d,1}$ with the mostly minus metric $\eta = \text{diag}(1,-1,\dots,-1)$. Following the prescription of Ref. [35], a bosonic SPT phase in this dimension is described by an $O(d+2)$ NLSM with topological theta term where the coefficient of the theta term is given by $\theta = 2\pi k$ with $k \in \mathbb{Z}$. The $O(d+2)$ NLSM is a theory of a $(d+2)$-component unit vector field $\mathbf{n}$ (i.e., $\mathbf{n}\cdot\mathbf{n} = 1$) with components n_a, $a = 1,\dots,d+2$. Because of the constraint, the configuration space (or target space) of the NLSM field is the $d+1$-dimensional sphere S^{d+1}. Latin indices $a, b, c, \dots$, which label components of n_a, can be raised and lowered with the Euclidean metrics δ^{ab}, δ_{ab}, and so n^a and n_a are numerically equal to each other. In what follows we use the summation convention for any indices (Latin or Greek) which appear once in an upper position and once in a lower position in any expression.

The NLSM action describing the SPT phase is

$$S_{bulk}[\mathbf{n}] = \int d^{d+1}x\ \frac{1}{2g}(\partial^\mu n^a)(\partial_\mu n_a) + S_\theta[\mathbf{n}]\ , \tag{4.3.5}$$

where $g > 0$ is the coupling constant of the NLSM (with units of (length)$^{d-1}$), and $S_\theta[\mathbf{n}]$ is the theta term. To write the theta term in a compact way we first introduce some notation. Let ω_{d+1} be the volume form on S^{d+1}. Explicitly, we have

$$\omega_{d+1} = \sum_{a=1}^{d+2}(-1)^{a-1} n_a dn_1 \wedge \cdots \wedge \overline{dn_a} \wedge \cdots \wedge dn_{d+2}\ , \tag{4.3.6}$$

where the overline means to omit that term from the wedge product. We also use the notation $\mathcal{A}_{d+1} \equiv \text{Area}[S^{d+1}] = \frac{2\pi^{\frac{d+2}{2}}}{\Gamma(\frac{d+2}{2})}$ for the area of the sphere S^{d+1}. In terms of these quantities, the theta term can be written compactly in differential form notation as

$$S_\theta[\mathbf{n}] = \frac{\theta}{\mathcal{A}_{d+1}}\int_{\mathbb{R}^{d,1}} \mathbf{n}^*\omega_{d+1}\ , \tag{4.3.7}$$

where $\mathbf{n}^*\omega_{d+1}$ denotes the pull-back to spacetime of the form ω_{d+1} via the map $\mathbf{n} : \mathbb{R}^{d,1} \to S^{d+1}$. In coordinates this becomes

$$S_\theta[\mathbf{n}] = \frac{\theta}{\mathcal{A}_{d+1}}\int d^{d+1}x\ \epsilon^{a_1\cdots a_{d+2}} n_{a_1}\partial_{x^0}n_{a_2}\partial_{x^1}n_{a_3}\cdots\partial_{x^d}n_{a_{d+2}}\ . \tag{4.3.8}$$

For the description of SPT phases we have $\theta = 2\pi k$ for integer k. The reason for choosing $\theta = 2\pi k$ is that at these values of θ the NLSM is expected to flow to a disordered ($g \to \infty$) fixed point under the Renormalization Group[35]. In addition we note that the full action of Eq. (4.3.5) (including theta term) has an $SO(d+2)$ global symmetry, where the action of the group on the NLSM field is given by $n_a \to {R_a}^b n_b$ for any matrix $R \in SO(d+2)$. When the coefficient θ is set to zero this symmetry is promoted to an $O(d+2)$ global symmetry (under a general transformation $R \in O(d+2)$ the theta term transforms only by acquiring the sign det$[R] = \pm 1$). The fixed point theory (with $g \to \infty$ at $\theta = 2\pi k$) is gapped and has a unique ground state which does not break the $SO(d+2)$ symmetry of the NLSM with theta term[160]. This property of the disordered ground state of the NLSM at $\theta = 2\pi k$ is one of the main reasons why these field theories are useful for describing SPT phases.

SPT phases are classified according to their symmetry group G. In the NLSM description of Ref. [35] this symmetry is encoded in a homomorphism $\sigma : G \to O(d+2)$, which maps $g \in G$ to some $(d+2) \times (d+2)$ matrix $\sigma(g) \in O(d+2)$. We refer to such a σ as a *symmetry assignment*. According to the NLSM classification of SPT phases, if $g \in G$ represents an *internal* unitary symmetry operation (i.e., g does not have any action on the spacetime coordinates) then σ should be chosen so that det$[\sigma(g)] = 1$. In this case it is then clear that the action of g leaves the theta term invariant. On the other hand, if $g \in G$ represents the time-reversal operation, then σ should be chosen so that det$[\sigma(g)] = -1$. Since the time-reversal operation also sends $t \to -t$ (in addition to its action on the components of the NLSM field), the minus sign in the theta term from det$[\sigma(g)]$ will be canceled by the minus sign from sending $\partial_t \to -\partial_t$. Thus, choosing det$[\sigma(g)] = -1$ in this case ensures that the theta term is invariant under the time-reversal transformation.

Not all NLSMs with a symmetry assignment will describe a non-trivial SPT phase. For example an NLSM with a symmetry assignment σ will describe a trivial phase if there exists a vector $\mathbf{v}$ such that $\sigma(g)\mathbf{v} = \mathbf{v}\ \forall g \in G$. This is because in this case we are allowed to add a term $\mathbf{n} \cdot \mathbf{v}$ to the NLSM action without breaking the symmetry of the group G. Such a term will then drive the system into a trivial phase in which $\mathbf{n}$ is parallel or anti-parallel to $\mathbf{v}$ at all points in space. If a vector $\mathbf{v}$ with this property does not exist, then the NLSM with symmetry assignment σ can describe a non-trivial SPT phase.

When an SPT phase has a bulk description in terms of an $O(d+2)$ NLSM with theta term and theta angle $\theta = 2\pi k$, its d-dimensional boundary is described by an $O(d+2)$ NLSM with Wess-Zumino (WZ) term at level k. Let us for simplicity study the boundary perpendicular to the x^d direction, so on the boundary we have coordinates x^μ, $\mu = 0, \dots, d-1$, and the boundary spacetime is $\mathbb{R}^{d-1,1}$. To write down the WZ term we need to extend the field configuration n_a into a fictitious extra dimension of the boundary spacetime. We take $s \in [0,1]$ to be the coordinate for this extra direction, and define $\mathcal{B} = [0,1] \times \mathbb{R}^{d-1,1}$ to be the extended boundary spacetime. Let $\tilde{n}_a(x^\mu, s)$ be an extension of the field n_a into the s direction. It is typical to choose boundary conditions in the extra direction so that $\tilde{n}_a(x^\mu, 1) = \delta_{a,1}$ (i.e., a trivial configuration) and $\tilde{n}_a(x^\mu, 0) = n_a(x^\mu)$ so that the physical boundary spacetime is located at $s = 0$. Then the action for the boundary theory takes the form

$$S_{bdy}[\mathbf{n}] = \int d^d x\ \frac{1}{2g_{bdy}}(\partial^\mu n^a)(\partial_\mu n_a) + S_{WZ}[\mathbf{n}]\ , \tag{4.3.9}$$

where the WZ term is

$$S_{WZ}[\mathbf{n}] = \frac{2\pi k}{\mathcal{A}_{d+1}} \int_{\mathcal{B}} \tilde{\mathbf{n}}^* \omega_{d+1} \ . \tag{4.3.10}$$

Here g_{bdy} is the coupling constant for the boundary theory, and the WZ term now involves the pull-back of ω_{d+1} to $\mathcal{B}$ (the extended boundary spacetime) via the map $\tilde{\mathbf{n}} : \mathcal{B} \to S^{d+1}$. Again, in coordinates this takes the form

$$S_{WZ}[\mathbf{n}] = \frac{2\pi k}{\mathcal{A}_{d+1}} \int_0^1 ds \int d^d x \ \epsilon^{a_1 \cdots a_{d+2}} \tilde{n}_{a_1} \partial_s \tilde{n}_{a_2} \partial_{x^0} \tilde{n}_{a_3} \cdots \partial_{x^{d-1}} \tilde{n}_{a_{d+2}} \ . \tag{4.3.11}$$

We now discuss the specific symmetry assignments $\sigma : G \to O(d+2)$ which will be used to construct NLSM descriptions of BIQH states in odd spacetime dimensions and BTI states in even spacetime dimensions. We start with the case of BIQH states in $2m-1$ spacetime dimensions. In this case the integer m is related to d by the relation $2m = d+2$, and the BIQH state is described by an $O(2m)$ NLSM with theta term. In the BIQH case the symmetry group is just $G = U(1)$ and the particular $U(1)$ symmetry that we are interested in is embedded in the full $O(2m)$ group as follows. We first combine pairs of the $2m$ components n_a of the NLSM field to create the m boson fields

$$b_\ell = n_{2\ell-1} + i n_{2\ell} \ , \ \ell = 1, \dots, m \ . \tag{4.3.12}$$

Then the $U(1)$ symmetry we consider acts on the NLSM field as

$$U(1) : b_\ell \to e^{i\xi} b_\ell, \forall \ell \ , \tag{4.3.13}$$

where ξ is a constant parameter. We can consider the fields b_ℓ to be m complex scalar fields of charge 1, but subject to the constraint $\sum_{\ell=1}^m |b_\ell|^2 = 1$, which is equivalent to the constraint $\mathbf{n} \cdot \mathbf{n} = 1$ for the NLSM field n_a. This choice of $U(1)$ transformation, and the corresponding pairing of the components of $\mathbf{n}$ into the bosons b_ℓ, is convenient, but it is not unique. Since the NLSM action with theta term (or WZ term) is still invariant under the group $SO(2m)$, we can do any change of basis $n^a \to M_a{}^b n_b$ with $M \in SO(2m)$ to obtain a theory with a different action of the $U(1)$ symmetry, but with the same physical properties. As discussed above, the most important property of the symmetry assignment is that there should not be any vector $\mathbf{v}$ that remains fixed under the $U(1)$ action. Indeed, if such a $\mathbf{v}$ exists then the NLSM with this symmetry assignment describes a trivial phase. The choice above satisfies this requirement.

For the case of BTI states in even dimensions $2m$, the integer m is instead related to d by the formula $2m+1 = d+2$, so that these states are described by $O(2m+1)$ NLSMs with theta term. As we discussed in the previous subsection the symmetry group in this case is $G = U(1) \rtimes \mathbb{Z}_2^T$ for m even and $G = U(1) \rtimes \mathbb{Z}_2^C$ for m odd. To define the symmetry assignment σ in this case we again take pairs of the first $2m$ components of the NLSM field and combine them into bosons b_ℓ, $\ell = 1, \dots, m$ as done for the BIQH case. The $U(1)$ symmetry we consider again acts as in Eq. (4.3.13) on these bosons, but leaves the final component n_{2m+1} of the NLSM field fixed. Finally, in the BTI case the additional discrete $\mathbb{Z}_2$ symmetry (which is either $\mathbb{Z}_2^T$ or $\mathbb{Z}_2^C$ depending on the parity of m) is taken to act on the NLSM field as

$$\mathbb{Z}_2 : n_a \quad \rightarrow \quad n_a \ , \ a = 1, 3, \dots, 2m-1, \tag{4.3.14a}$$

$$n_a \quad \rightarrow \quad -n_a \ , \ a = 2, 4, \dots, 2m, 2m+1 \ . \tag{4.3.14b}$$

In the case where the $\mathbb{Z}_2$ symmetry is time-reversal $\mathbb{Z}_2^T$, we also need to send $t \to -t$ in the argument of n_a and in the action. Under the transformation in Eq. (4.3.14), the theta term of the NLSM picks up the sign $(-1)^{m+1}$. So we see that for m odd the theta term in the NLSM automatically has this symmetry, while in the case of m even it must be supplemented with the replacement $t \to -t$, which gives an extra minus sign in the theta term. So the NLSM has the internal, unitary $\mathbb{Z}_2^C$ particle-hole symmetry in the case of m odd, while in the case of m even it has the anti-unitary time-reversal symmetry $\mathbb{Z}_2^T$.

Now that we know how the fields in the NLSM description transform under the $U(1)$ symmetry of the BIQH and BTI phases, we can considering coupling the NLSM theory, and in particular the boundary theory which involves a WZ term, to the external electromagnetic field $A = A_\mu dx^\mu$. In order to do this, we are going to need the tool of gauged WZ actions.

4.3.3 Gauged Wess-Zumino actions

We now give a discussion of the theory of gauged WZ actions, mostly focusing on the general philosophy behind the construction of a gauged WZ action. The details of this construction will be worked out explicitly for the boundary theories of the BIQH and BTI phases in all dimensions in later sections of this Chapter. In addition, in Appendix C.1 we review the relation between gauged WZ actions and equivariant cohomology, and we re-examine the gauged WZ actions constructed in this Chapter from this more mathematical point of view.

Before we start, let us note that the kinetic term for the NLSM is easily gauged using ordinary minimal coupling (also known as a "Peierls substitution" in a condensed matter context). In fact, the gauged kinetic term is most simply written in terms of the b_ℓ as

$$S_{kin,gauged}[\mathbf{n}, A] = \int d^d x \, \frac{1}{2g_{bdy}} \sum_{\ell=1}^{m} (D^\mu b_\ell)^* (D_\mu b_\ell) \ , \tag{4.3.15}$$

for the boundary of the BIQH state ($d+2=2m$), or

$$S_{kin,gauged}[\mathbf{n},A]=\int d^dx\,\frac{1}{2g_{bdy}}\left[\sum_{\ell=1}^{m}(D^\mu b_\ell)^*(D_\mu b_\ell)+(\partial^\mu n_{2m+1})(\partial_\mu n_{2m+1})\right], \tag{4.3.16}$$

for the boundary of a BTI state ($d+2=2m+1$), where $D_\mu=\partial_\mu-iA_\mu$ is the usual covariant derivative. Note here that since we are only interested in enforcing a $U(1)$ subroup of the full $SO(d+2)$ symmetry group of the NLSM, we could allow a different boundary coupling constant $g_{bdy,\ell}$ for each species b_ℓ of boson. This type of anisotropy in the coupling constant will not affect the results in the rest of the Chapter, since those results only depend on the form of the WZ term.

Gauging the WZ term is more subtle. The main problem we face in attempting to gauge this term is the fact that the WZ term is written as an integral of an expression involving the field $\tilde{n}_a$ over the $(d+1)$-dimensional extended spacetime $\mathcal{B}$. One method[41] for gauging a WZ term involves defining an extension $\tilde{A}$ of the gauge field A into the extra s-direction, and then applying the usual minimal coupling procedure (but using the extended field $\tilde{A}$) inside the WZ term. This has the effect of replacing the integrand $\tilde{\mathbf{n}}^*\omega_{d+1}$ of the WZ term in Eq. (4.3.10) with $\tilde{\mathbf{n}}^*\omega^{\tilde{A}}_{d+1}$, where $\omega^{\tilde{A}}_{d+1}$ represents the volume form on S^{d+1} but with the ordinary exterior derivative d replaced with a gauge-covariant exterior derivative D (the precise form of D is not important for the general discussion here). However, minimal coupling alone is not sufficient, as varying the minimally-coupled WZ action does not lead to d-dimensional equations of motion, i.e., the resulting equations of motion depend on the extensions $\tilde{n}_a$ and $\tilde{A}$. To remedy this the authors of Ref. [41] used the following prescription. They suggested that one should add a second term $U(\tilde{n}_a,\tilde{A})$ to the integrand of the WZ term such that the combination $\omega^{\tilde{A}}_{d+1}+U$ is a closed form on the extended spacetime. Since a closed form is locally exact (i.e., a closed form ω can be written as $\omega=d\gamma_i$ for some γ_i on each coordinate patch $\mathcal{U}_i$ of the manifold), variation of this new WZ term leads to d-dimensional equations of motion on each coordinate patch of the original spacetime manifold. There is, however, one conceptual issue with this method, which the authors of Ref. [41] point out (see their discussion in the paragraph after equation 4.7). The problem is that in the usual setup of the WZ term, the form $\omega^{\tilde{A}}_{d+1}$ (and also ω_{d+1}) is a $(d+1)$-form on the $(d+1)$-dimensional extended spacetime $\mathcal{B}$, and so it is trivially closed. Therefore in order to apply the method of Ref. [41] one has to imagine that the extended spacetime $\mathcal{B}$ is embedded in a spacetime $\mathcal{X}$ of even higher dimension so that $d\omega^{\tilde{A}}_{d+1}$ is not trivially equal to zero.

From this discussion it is clear that gauging a WZ is in general a difficult procedure. However, for the problems encountered in this Chapter, in which we only deal with a $U(1)$ subgroup of the full $O(d+2)$ symmetry of the NLSM theories, we do not need the complicated machinery developed in Ref. [41]. Instead, we use the following concrete procedure (which is similar in spirit to the methods used in Refs. [39, 40]) to gauge the $U(1)$ symmetry of our theories. First we consider how the WZ term changes under the transformation $b_\ell\to e^{i\xi}b_\ell$ (with a spacetime-dependent ξ). We will see that it changes by a term which is a total derivative, which means that the change of the WZ term can be written as an integral only over the physical boundary spacetime $\mathbb{R}^{d-1,1}$ instead of over the extended spacetime $\mathcal{B}$. Next we attempt to cancel this change in the action by adding an integral over spacetime of the NLSM field coupled to A. We will see that this procedure usually needs to be iterated several times because the counterterms that we add to the action may not transform nicely under a gauge transformation, where "nicely" is defined below by Eq. (4.3.17). We use the following criterion, inspired by the discussion in Ref. [39], for determining when the action has been properly gauged.

Gauging principle: The correctly gauged action $S_{gauged}[\mathbf{n}, A]$, if it is not completely gauge-invariant, must transform under a gauge transformation $b_\ell \to e^{i\xi} b_\ell$, $A \to A + d\xi$, as

$$S_{gauged}[\mathbf{n}, A] \to S_{gauged}[\mathbf{n}, A] + \delta_\xi S_{gauged}[A, \xi] \ , \tag{4.3.17}$$

where we have used the notation $\delta_\xi S_{gauged}$ to indicate the change in S_{gauged} under a gauge transformation. The key point here is that the change in the action under a gauge transformation depends only on A and ξ, but not on the matter field $\mathbf{n}$.

Let us also note here that in this Chapter we use the word "anomaly" to refer to the change in the action (or action plus path integral measure) under a $U(1)$ gauge transformation. There is no anomaly if the action (plus path integral measure) is gauge-invariant. The gauging principle stated above then simply asserts that the anomaly $\delta_\xi S_{gauged}[A, \xi]$ of the gauged action $S_{gauged}[\mathbf{n}, A]$ should only depend on A and ξ.

We will see in the following sections that we may need to add several counterterms to the WZ action to get Eq. (4.3.17) to hold. In the BIQH case the correctly gauged action still transforms under a gauge transformation, and so the $U(1)$ symmetry of the boundary theory of the BIQH phase is anomalous. This fact is what allows us to deduce the bulk CS response of the BIQH state. On the other hand, for the surface of the BTI it is possible to construct a completely gauge-invariant action. However, from the form of the gauge-invariant action we will be able to see that if the NLSM field condenses in a way that preserves the $U(1)$ symmetry, but breaks the $\mathbb{Z}_2$ symmetry of the BTI phase, then the surface of the BTI will exhibit a $\mathbb{Z}_2$ symmetry-breaking Quantum Hall response.

4.4 Electromagnetic response of BIQH states in all odd dimensions

In this section we construct the gauged WZ action for the boundary of BIQH states in all odd dimensions. The action we construct satisfies the gauging principle of Eq. (4.3.17), but is still not completely gauge-invariant, as evidenced in the $U(1)$ anomaly of the boundary theory of the BIQH state. We then use the $U(1)$ anomaly of the gauged boundary action to calculate the bulk CS response of the BIQH state in all odd dimensions. As we discussed in the introduction, we find that for the BIQH state in $2m-1$ dimensions the level N_{2m-1} of the CS term appearing in the effective action is quantized in units of $m!$. We then give a more intuitive derivation of the BIQH response using only the dimensional reduction properties of CS terms and of theta terms in NLSMs. This second derivation relies on results which we derive in Appendices C.4 and C.5. This intuitive picture confirms our more technical derivation using gauged WZ actions.

The result in this section is related to the results of several other sections of this Chapter. In the next section, Sec. 4.5, we show that the factor of $m!$ for the CS response of the BIQH state computed in this section can be understood by requiring that partition functions containing the CS response action be invariant under large $U(1)$ gauge transformations on general Euclidean manifolds. Later, in Appendix C.1, we re-examine the gauged WZ action constructed in this section in light of the well-known connection between gauged WZ actions and equivariant cohomology of the target space of the NLSM. The construction of a gauged WZ action for the boundary of the BIQH state is equivalent to the problem of constructing an *equivariant extension* (with respect to the $U(1)$ symmetry) of the volume form ω_{2m-1} for S^{2m-1}. In Appendix C.1 we attempt to construct such an extension, and then show that the construction fails at the last step. The fact that such an extension does not exist is mathematically equivalent to our finding that the gauged action for the boundary of the BIQH state still has a $U(1)$ anomaly. In Appendix C.1 we also show that the differential forms $\Omega^{(r)}$, which appear later in this section in the counterterms of Eq. (4.4.29), are the same forms which appear in the construction of the equivariant extension of ω_{2m-1} (although the extension fails at the last step in this case as mentioned above).

Let us make a few remarks on the notation used in this section and in later sections of the Chapter. In what follows we omit the pull-back symbol $\mathbf{n}^*$ so as not to clutter the notation, but one should always remember that the integrand of any integral should be pulled back to spacetime (or the extended spacetime, in which case one would write $\tilde{\mathbf{n}}^*$). In addition we will express many quantities in terms of the integer m instead of d. Recall that these are related by $2m = d + 2$ in the BIQH case. So for example we write the WZ term as

$$S_{WZ}[\mathbf{n}] = \frac{2\pi k}{\mathcal{A}_{2m-1}} \int_{\mathcal{B}} \omega_{2m-1} \ . \tag{4.4.1}$$

For later use we also define several differential forms which are constructed from the components of the NLSM field. We define the one form $\mathcal{J}_\ell$ and two form $\mathcal{K}_\ell$ by

$$\mathcal{J}_\ell = n_{2\ell-1} dn_{2\ell} - n_{2\ell} dn_{2\ell-1} \tag{4.4.2a}$$

$$\mathcal{K}_\ell = dn_{2\ell-1} \wedge dn_{2\ell} \ . \tag{4.4.2b}$$

Under a gauge transformation $b_\ell \to e^{i\xi} b_\ell$ these forms transform as

$$\mathcal{J}_\ell \;\to\; \mathcal{J}_\ell + (n_{2\ell-1}^2 + n_{2\ell}^2) d\xi \tag{4.4.3a}$$

$$\mathcal{K}_\ell \;\to\; \mathcal{K}_\ell + (n_{2\ell-1} dn_{2\ell-1} + n_{2\ell} dn_{2\ell}) \wedge d\xi \ . \tag{4.4.3b}$$

We also note here that

$$\mathcal{K}_\ell = \frac{1}{2} d\mathcal{J}_\ell \ , \tag{4.4.4}$$

and so

$$d\mathcal{K}_\ell = 0 \ , \tag{4.4.5}$$

i.e., $\mathcal{K}_\ell$ is an *exact* differential form.

4.4.1 $O(4)$ NLSM with WZ term in two spacetime dimensions

Before presenting the gauged action for any integer m, we warm up with an explicit calculation for the simplest possible case, which is the $O(4)$ NLSM with WZ term which appears at the two-dimensional boundary of the BIQH state in three dimensions. We also mention here that an $O(4)$ NLSM with WZ term in two dimensions is equivalent to a model of an $SU(2)$ matrix field $U = n_4 \mathbb{I} + \sum_{a=1}^{3} n_a \sigma^a$ (where σ^a are the three Pauli matrices) with WZ term for U, so the analysis in this subsection is actually a special case of the analysis done in Refs. [39, 40]. Although we focus on the case of a continuous symmetry (namely the $U(1)$ charge conservation symmetry), we also note here that anomalies in the two-dimensional boundary theories of SPT phases protected by the symmetry of a *finite* abelian group were considered previously in Ref. [192].

In the $O(4)$ case the volume form can be written as

$$\omega_3 = \mathcal{J}_1 \wedge \mathcal{K}_2 + \mathcal{J}_2 \wedge \mathcal{K}_1 \ . \tag{4.4.6}$$

Under the transformation $b_\ell \to e^{i\xi} b_\ell$ we have

$$\begin{aligned} \delta_\xi \omega_3 &= \mathcal{K}_1 \wedge d\xi + \mathcal{K}_2 \wedge d\xi \\ &= \frac{1}{2} d\mathcal{J}_1 \wedge d\xi + \frac{1}{2} d\mathcal{J}_2 \wedge d\xi \\ &= \frac{1}{2} d \left[\mathcal{J}_1 \wedge d\xi + \mathcal{J}_2 \wedge d\xi \right] \ , \end{aligned} \tag{4.4.7}$$

which is a total derivative. So we find (neglecting any terms coming from the boundary of the physical spacetime

$\mathbb{R}^{1,1}$)

$$\delta_\xi S_{WZ}[\mathbf{n}] = \frac{2\pi k}{\mathcal{A}_3}\frac{1}{2}\int_{\mathbb{R}^{1,1}} (\mathcal{J}_1 + \mathcal{J}_2) \wedge d\xi \ . \tag{4.4.8}$$

We attempt to cancel this variation by adding the counterterm

$$S^{(1)}_{ct}[\mathbf{n}, A] = -\frac{2\pi k}{\mathcal{A}_3}\frac{1}{2}\int_{\mathbb{R}^{1,1}} (\mathcal{J}_1 + \mathcal{J}_2) \wedge A \ . \tag{4.4.9}$$

It is clear that when we send $A \to A + d\xi$ in $S^{(1)}_{ct}$ it will cancel the gauge variation of the WZ term.

At this point our candidate for the gauged WZ term is then

$$S_{WZ,gauged}[\mathbf{n}, A] = S_{WZ}[\mathbf{n}] + S^{(1)}_{ct}[\mathbf{n}, A] \ . \tag{4.4.10}$$

However, this action is not completely gauge-invariant, and under a gauge transformation we find

$$\begin{aligned} \delta_\xi S_{WZ,gauged}[\mathbf{n}, A] &= -\frac{2\pi k}{\mathcal{A}_3}\frac{1}{2}\int_{\mathbb{R}^{1,1}} (\delta_\xi\mathcal{J}_1 + \delta_\xi\mathcal{J}_2) \wedge A \\ &= -\frac{2\pi k}{\mathcal{A}_3}\frac{1}{2}\int_{\mathbb{R}^{1,1}} d\xi \wedge A \\ &= -\frac{k}{2\pi}\int_{\mathbb{R}^{1,1}} d\xi \wedge A \\ &= k\int_{\mathbb{R}^{1,1}} \xi\left(\frac{F}{2\pi}\right) \ , \end{aligned} \tag{4.4.11}$$

where we used the formula for $\delta_\xi \mathcal{J}_\ell$ from Eq. (4.4.3), the fact that $\mathbf{n}$ is a unit vector field, $\mathcal{A}_3 = 2\pi^2$, and also performed an integration by parts in the last line ($F = dA$). We conclude that the $U(1)$ symmetry here is anomalous and, since the kinetic term has been made completely gauge-invariant, the total anomaly of the boundary theory is given by Eq. (4.4.11). We also note that the anomaly in Eq. (4.4.11) is exactly what is needed to cancel the gauge variation of the bulk CS action of Eq. (4.3.1) with $N_3 = -2k$.

4.4.2 The $O(2m)$ NLSM with WZ term in $2m - 2$ spacetime dimensions

Now we move on to the general case of an $O(2m)$ NLSM with WZ term on the $2m - 2$ dimensional boundary of a BIQH state in $2m - 1$ dimensions (recall that m is related to the integer d in the BIQH case by $d = 2m - 2$, so that d is also the dimension of the boundary spacetime). In this case we find that a total of $m - 1$ counterterms are needed in order for the gauged WZ action to transform as in Eq. (4.3.17) under a gauge transformation. To start we note that the volume form ω_{2m-1} can be re-written using the forms $\mathcal{J}_\ell$ and $\mathcal{K}_\ell$ as

$$\omega_{2m-1} = \frac{1}{(m-1)!} \sum_{\ell_1,\ldots,\ell_m=1}^{m} \mathcal{J}_{\ell_1} \wedge \mathcal{K}_{\ell_2} \wedge \cdots \wedge \mathcal{K}_{\ell_m} \ . \tag{4.4.12}$$

To see it, simply note that if any of $\ell_2, \ldots, \ell_m$ are equal to each other or to ℓ_1 then the wedge product vanishes. So each index ℓ_s can be summed over the full range of 1 to m. However, this means that we are actually over-counting in the sum over all ℓ_s. This is not a problem though as $\mathcal{K}_{\ell_s}$ can be commuted past each other in the wedge products (they are all two-forms), so all we need to do to remedy this is to divide by the factor of $(m-1)!$, where $m-1$ is the number of factors of $\mathcal{K}_\ell$ appearing in the expression.

Now for any integer r in the range $0, \ldots, m-1$, we introduce the form

$$\Omega^{(r)} = \sum_{\ell_1,\ldots,\ell_{m-r}=1}^{m} \mathcal{J}_{\ell_1} \wedge \mathcal{K}_{\ell_2} \wedge \cdots \wedge \mathcal{K}_{\ell_{m-r}} \ . \tag{4.4.13}$$

In particular, we have $\omega_{2m-1} = \frac{1}{(m-1)!}\Omega^{(0)}$ and $\Omega^{(m-1)} = \sum_{\ell_1=1}^{m} \mathcal{J}_{\ell_1}$. In Appendix C.1 we give a mathematical interpretation of these forms in terms of $U(1)$-equivariant cohomology of S^{2m-1}. The following formula for the change in $\Omega^{(r)}$ under a gauge transformation is the essential ingredient in our construction of the full gauged WZ action.

Claim: Under a gauge transformation $b_\ell \to e^{i\xi} b_\ell$ we have $\Omega^{(r)} \to \Omega^{(r)} + \delta_\xi \Omega^{(r)}$ with

$$\delta_\xi \Omega^{(r)} = \frac{1}{2} d\Omega^{(r+1)} \wedge d\xi \ . \tag{4.4.14}$$

Proof: Using Eqs. (4.4.3) we can show

$$\begin{aligned} \delta_\xi \Omega^{(r)} &= \sum_{\ell_1,\ldots,\ell_{m-r}=1}^{m} (n_{2\ell_1-1}^2 + n_{2\ell_1}^2) \mathcal{K}_{\ell_2} \wedge \cdots \wedge \mathcal{K}_{\ell_{m-r}} \wedge d\xi \\ &+ \sum_{s=2}^{m-r} \sum_{\ell_1,\ldots,\ell_{m-r}=1}^{m} \mathcal{J}_{\ell_1} \wedge \mathcal{K}_{\ell_2} \wedge \cdots \wedge \overline{\mathcal{K}_{\ell_s}} \wedge \cdots \wedge \mathcal{K}_{\ell_{m-r}} \wedge (n_{2\ell_s-1} dn_{2\ell_s-1} + n_{2\ell_s} dn_{2\ell_s}) \wedge d\xi \ , \end{aligned} \tag{4.4.15}$$

where the overline again means to omit that term from the wedge product. Next we use the two properties

$$\sum_{\ell=1}^{m} (n_{2\ell-1}^2 + n_{2\ell}^2) = 1 \tag{4.4.16a}$$

$$\sum_{\ell=1}^{m} (n_{2\ell-1} dn_{2\ell-1} + n_{2\ell} dn_{2\ell}) = 0 \ , \tag{4.4.16b}$$

which follow from the fact that $\mathbf{n}$ is a unit vector field with $2m$ components, to find that

$$\delta_\xi \Omega^{(r)} = \sum_{\ell_2,\ldots,\ell_{m-r}=1}^{m} \mathcal{K}_{\ell_2} \wedge \cdots \wedge \mathcal{K}_{\ell_{m-r}} \wedge d\xi \ , \tag{4.4.17}$$

or after re-indexing,

$$\delta_\xi \Omega^{(r)} = \sum_{\ell_1,\ldots,\ell_{m-(r+1)}=1}^{m} \mathcal{K}_{\ell_1} \wedge \cdots \wedge \mathcal{K}_{\ell_{m-(r+1)}} \wedge d\xi \ . \tag{4.4.18}$$

So in fact, only the term in the first line of Eq. (4.4.15) has contributed. Next we write $\mathcal{K}_{\ell_1} = \frac{1}{2} d\mathcal{J}_{\ell_1}$ and use the fact that $\mathcal{K}_\ell$ is closed to find

$$\begin{aligned} \delta_\xi \Omega^{(r)} &= \frac{1}{2} \sum_{\ell_1,\ldots,\ell_{m-(r+1)}=1}^{m} d\mathcal{J}_{\ell_1} \wedge \mathcal{K}_{\ell_2} \wedge \cdots \wedge \mathcal{K}_{\ell_{m-(r+1)}} \wedge d\xi \\ &= \frac{1}{2} d\Omega^{(r+1)} \wedge d\xi \ , \end{aligned} \tag{4.4.19}$$

which completes the proof. ■

With Eq. (4.4.14) in hand we can now construct the properly gauged action step by step. We go through the first few steps explicitly, and then write down the final answer. To start, the change of the WZ term under a gauge transformation is

$$\begin{aligned} \delta_\xi S_{WZ}[\mathbf{n}] =& \frac{2\pi k}{\mathcal{A}_{2m-1}} \frac{1}{(m-1)!} \int_{\mathcal{B}} \delta_\xi \Omega^{(0)} \\ =& \frac{2\pi k}{\mathcal{A}_{2m-1}} \frac{1}{(m-1)!} \frac{1}{2} \int_{\mathcal{B}} d\Omega^{(1)} \wedge d\xi \\ =& \frac{2\pi k}{\mathcal{A}_{2m-1}} \frac{1}{(m-1)!} \frac{1}{2} \int_{\mathbb{R}^{d-1,1}} \Omega^{(1)} \wedge d\xi \ . \end{aligned} \tag{4.4.20}$$

So the first counterterm we should add is

$$S_{ct}^{(1)}[\mathbf{n}, A] = -\frac{2\pi k}{\mathcal{A}_{2m-1}} \frac{1}{(m-1)!} \frac{1}{2} \int_{\mathbb{R}^{d-1,1}} \Omega^{(1)} \wedge A \ . \tag{4.4.21}$$

The part of the action containing the WZ term is now

$$S'_{WZ,gauged}[\mathbf{n}, A] = S_{WZ}[\mathbf{n}] + S_{ct}^{(1)}[\mathbf{n}, A] \ , \tag{4.4.22}$$

and under a gauge transformation we find

$$\delta_\xi S'_{WZ,gauged}[\mathbf{n}, A] = -\frac{2\pi k}{\mathcal{A}_{2m-1}} \frac{1}{(m-1)!} \frac{1}{2} \int_{\mathbb{R}^{d-1,1}} \delta_\xi \Omega^{(1)} \wedge A \ , \tag{4.4.23}$$

which becomes

$$\delta_\xi S'_{WZ,gauged}[\mathbf{n},A] = -\frac{2\pi k}{\mathcal{A}_{2m-1}}\frac{1}{(m-1)!}\frac{1}{2^2}\int_{\mathbb{R}^{d-1,1}} d\Omega^{(2)}\wedge d\xi\wedge A\ . \tag{4.4.24}$$

Now we note that

$$d\left(\Omega^{(2)}\wedge d\xi\wedge A\right) = d\Omega^{(2)}\wedge d\xi\wedge A + \Omega^{(2)}\wedge d\xi\wedge F\ , \tag{4.4.25}$$

and we use this to do an integration by parts. Neglecting boundary terms (in general we neglect all terms coming from the boundaries of the physical boundary spacetime), we now have

$$\delta_\xi S'_{WZ,gauged}[\mathbf{n},A] = \frac{2\pi k}{\mathcal{A}_{2m-1}}\frac{1}{(m-1)!}\frac{1}{2^2}\int_{\mathbb{R}^{d-1,1}} \Omega^{(2)}\wedge d\xi\wedge F\ . \tag{4.4.26}$$

Therefore we should choose the second counterterm to be

$$S^{(2)}_{ct}[\mathbf{n},A] = -\frac{2\pi k}{\mathcal{A}_{2m-1}}\frac{1}{(m-1)!}\frac{1}{2^2}\int_{\mathbb{R}^{d-1,1}} \Omega^{(2)}\wedge A\wedge F\ , \tag{4.4.27}$$

and the total gauged action is now

$$S''_{WZ,gauged}[\mathbf{n},A] = S_{WZ}[\mathbf{n}] + S^{(1)}_{ct}[\mathbf{n},A] + S^{(2)}_{ct}[\mathbf{n},A]\ . \tag{4.4.28}$$

At this point the pattern is clear. After iterating this procedure we find that a total of $m-1$ counterterms are needed to construct a gauged WZ action which satisfies Eq. (4.3.17). The r^{th} counterterm (for $r = 1,\ldots,m-1$) is given by

$$S^{(r)}_{ct}[\mathbf{n},A] = -\frac{2\pi k}{\mathcal{A}_{2m-1}}\frac{1}{(m-1)!}\frac{1}{2^r}\int_{\mathbb{R}^{d-1,1}} \Omega^{(r)}\wedge A\wedge F^{r-1}\ , \tag{4.4.29}$$

where F^{r-1} is shorthand for the wedge product of F with itself $r-1$ times. The total gauged action is then

$$S_{WZ,gauged}[\mathbf{n},A] = S_{WZ}[\mathbf{n}] + \sum_{r=1}^{m-1} S^{(r)}_{ct}[\mathbf{n},A]\ . \tag{4.4.30}$$

In Appendix C.1 we discuss this gauged WZ action from the point of view of $U(1)$-equivariant cohomology over the sphere S^{2m-1}.

When we look at the change of the full action $S_{WZ,gauged}[\mathbf{n},A]$ under a gauge transformation we find that it is not completely gauge-invariant. In other words, the $U(1)$ symmetry of the boundary theory of the BIQH state is anomalous, as we expect on physical grounds. The anomaly is controlled only by the final counterterm $S^{(m-1)}_{ct}[\mathbf{n},A]$, since all other contributions cancel by construction. Under a gauge transformation we have

$$\delta_\xi S_{WZ,gauged}[\mathbf{n}, A] = -\frac{2\pi k}{\mathcal{A}_{2m-1}} \frac{1}{(m-1)!} \frac{1}{2^{m-1}} \int_{\mathbb{R}^{d-1,1}} \delta_\xi \Omega^{(m-1)} \wedge A \wedge F^{m-2} . \tag{4.4.31}$$

Now we use $\delta_\xi \Omega^{(m-1)} = d\xi$, the formula $\mathcal{A}_{2m-1} = \frac{2\pi^m}{(m-1)!}$, and integrate by parts to arrive at the final formula

$$\delta_\xi S_{WZ,gauged}[\mathbf{n}, A] = k \int_{\mathbb{R}^{d-1,1}} \xi \left(\frac{F}{2\pi}\right)^{m-1} , \tag{4.4.32}$$

or in terms of the boundary spacetime dimension d,

$$\delta_\xi S_{WZ,gauged}[\mathbf{n}, A] = k \int_{\mathbb{R}^{d-1,1}} \xi \left(\frac{F}{2\pi}\right)^{\frac{d}{2}} . \tag{4.4.33}$$

4.4.3 Chern-Simons effective action for bulk electromagnetic response

We now use the result of the previous subsection to understand the bulk electromagnetic response of BIQH states in all odd spacetime dimensions. As we discussed in the Introduction, a Quantum Hall state in $2m-1$ dimensions is characterized by the presence of a CS term in the effective action $S_{eff}[A]$ for the electromagnetic field A. Recall that on $(2m-1)$-dimensional spacetime the CS term takes the form

$$S_{CS}[A] = \frac{N_{2m-1}}{(2\pi)^{m-1} m!} \int_{\mathcal{M}} A \wedge (dA)^{m-1} . \tag{4.4.34}$$

Now it is well known that under a gauge transformation $A \to A + d\xi$ the CS action changes by a boundary term,

$$\delta_\xi S_{CS}[A] = \frac{N_{2m-1}}{m!} \int_{\partial\mathcal{M}} \xi \left(\frac{F}{2\pi}\right)^{m-1} . \tag{4.4.35}$$

We can then deduce the coefficient N_{2m-1} for the bulk response of BIQH states by matching the variation of the bulk CS effective action for A with the anomaly of the boundary theory of the BIQH state (the $O(2m)$ NLSM with WZ term) which we calculated in the previous subsection. The gauge transformation of the bulk CS term must cancel the anomaly of the boundary theory in order for the entire system (bulk plus boundary) to be gauge-invariant. This is exactly the concept of anomaly inflow[75] which we mentioned in the introduction. Comparing Eq. (4.4.35) to Eq. (4.4.32) for the $U(1)$ anomaly of the $O(2m)$ theory with WZ term, we deduce that the coefficient N_{2m-1} must be given by

$$N_{2m-1} = -(m!)k, \ k \in \mathbb{Z} , \tag{4.4.36}$$

in order to cancel the anomaly of the boundary theory. Therefore we find that the level N_{2m-1} of the CS effective action for BIQH states in $2m-1$ spacetime dimensions is quantized in units of $m!$. This answer agrees with the known cases for three and five spacetime dimensions and gives a prediction for all odd dimensions beyond those. In Sec. 4.5 we discuss this peculiar quantization of the CS level from a mathematical point of view by studying the transformation of the CS term under large $U(1)$ gauge transformations on general Euclidean manifolds (including manifolds which do not admit a spin structure).

We also remark here that based on the form of the CS response for the BIQH state in $2m-1$ dimensions, we can conclude that the chiral anomaly of the boundary theory of the BIQH state is $m!$ times larger than the chiral anomaly of the boundary theory for a fermionic SPT phase in $2m-1$ dimensions with a bulk CS response at level one. So, for example, the anomaly of the boundary theory is twice as large when the bulk is three-dimensional ($m=2$ case) and six times as large when the bulk is five-dimensional ($m=3$).

4.4.4 A derivation of the response from the bulk physics

To close this section we present an alternative derivation of the response of the BIQH state. This derivation uses only bulk properties of the BIQH state, which should be contrasted with our derivation using gauged WZ actions which was based on the anomaly of the boundary theory. Recall again that the bulk of the BIQH state is described by an $O(2m)$ NLSM with theta term and theta angle $\theta=2\pi k$ (so we have a theta term and not a WZ term in the bulk description). The main reason for including this alternative derivation is that it provides a clear physical reason for the appearance of the $m!$ factor in the response. The derivation in this subsection uses only the dimensional reduction properties of the CS response action for the external field, and the theta term of the NLSM, which we now review.

We start by considering the CS response action at level N in $2m-1$ dimensions,

$$S_{CS}[A]=\frac{N}{(2\pi)^{m-1}m!}\int_{\mathbb{R}^{D,1}}A\wedge(dA)^{m-1}\ , \tag{4.4.37}$$

where D is the spatial dimension so that $D+1=2m-1$. Let $\mathbf{x}=(x^1,\dots,x^D)$ be the spatial coordinates. Now suppose we thread a delta function of 2π flux at a point $\mathbf{x}_0$ in the (x^{D-1},x^D) plane (i.e., $x_0^j=0$, $j=1,\dots,D-2$). Concretely, we set

$$F_{x^{D-1}x^D}\quad=\quad 2\pi\delta(x^{D-1}-x_0^{D-1})\delta(x^D-x_0^D)\ , \tag{4.4.38}$$

and we assume that $F_{x^jx^{D-1}}=F_{x^jx^D}=0\ \forall j=1,\dots,D-2$, and that $F_{x^jx^k}$ is independent of (x^{D-1},x^D) for

$j, k = 1, \dots, D-2$. Then, for this configuration, the CS response action reduces to

$$S_{CS}[A] \to \frac{N}{(2\pi)^{m-2}(m-1)!} \int_{\mathbb{R}^{D-2,1}} \tilde{A} \wedge (d\tilde{A})^{m-2} \,. \tag{4.4.39}$$

The key point is that it reduces to a CS term *at the same level* N on the $(D-2)$-dimensional space located at the point $\mathbf{x}_0$ in the (x^{D-1}, x^D) plane.

Now that we know what happens in the CS response action when we thread a 2π delta function flux of F in a particular plane, let us also see what happens in the NLSM description of the BIQH phase when this flux is inserted. In the NLSM description, the m bosons b_ℓ are all charged under the $U(1)$ symmetry. Therefore, threading a 2π delta function flux at the point $\mathbf{x}_0$ in the (x^{D-1}, x^D) plane will cause all of the bosons b_ℓ to have a vortex configuration in that plane around the point $\mathbf{x}_0$. By a vortex configuration we just mean that the phases of the complex numbers b_ℓ all wind by 2π as one encircles the point $\mathbf{x}_0$ in the (x^{D-1}, x^D) plane. So we conclude that threading a 2π delta function flux of F will create m vortex excitations in the $O(2m)$ NLSM which describes the bulk of the BIQH.

On the other hand, we are going to show that if a *single boson* b_ℓ for some ℓ has a vortex configuration at a point $\mathbf{x}_0$ in the (x^{D-1}, x^D) plane, then the $O(2m)$ NLSM action with $\theta = 2\pi k$ reduces to an $O(2m-2)$ NLSM with $\theta = 2\pi k$ living on the $(D-2)$-dimensional space at $\mathbf{x}_0$. So if we have a vortex in one boson only, then the NLSM theory for the BIQH state in $2m-1$ dimensions reduces to the NLSM theory for the BIQH state in $2m-3$ dimensions (inside the vortex core) and *with the same theta angle*.

We now prove the assertion in the previous paragraph that a vortex in one boson b_ℓ in the $O(2m)$ NLSM traps an $O(2m-2)$ NLSM with the same theta angle inside the vortex core. To do this we consider an explicit vortex ansatz for the NLSM field in which the last boson $b_m = n_{2m-1} + i n_{2m}$ takes on a vortex configuration. To set up the notation let (r, ϕ) be polar coordinates for the (x^{D-1}, x^D) plane, and let $\mathbf{y} = (x^1, \dots, x^D)$ be the coordinates for the remaining directions of space. Then our vortex ansatz has the form

$$\mathbf{n}(t, \mathbf{x}) = \{\sin(f(r))\mathbf{N}(t, \mathbf{y}), \cos(f(r))\mathbf{m}(\phi)\} \,. \tag{4.4.40}$$

where $\mathbf{N}(t, \mathbf{y})$ is a $(2m-2)$-component unit vector field depending only on t and $\mathbf{y}$, and $\mathbf{m}(\phi) = (\cos(\phi), \sin(\phi))$ represents the vortex configuration of the last two components of $\mathbf{n}$. The function $f(r)$ is assumed to satisfy the boundary conditions

$$f(0) = \frac{\pi}{2} \tag{4.4.41}$$

$$\lim_{r \to \infty} f(r) = 0 \,, \tag{4.4.42}$$

which means that the field $\mathbf{N}(t, \mathbf{y})$ lives in the core of the vortex. This vortex ansatz is equivalent to the $q = 1$, $n_q = 1$, case of the more general defect configurations for NLSMs considered in Appendix C.4. Using the dimensional reduction formula from Eq. (C.4.10) of Appendix C.4 we immediately derive that on this configuration the theta term of the $O(2m)$ NLSM reduces to

$$\begin{aligned} S_\theta[\mathbf{n}] &= \frac{\theta}{\mathcal{A}_{2m}} \int_{\mathbb{R}^{D,1}} \mathbf{n}^* \omega_{2m} \\ &\to \frac{\theta}{\mathcal{A}_{2m-2}} \int_{\mathbb{R}^{D-2,1}} \mathbf{N}^* \omega_{2m-2} \ . \end{aligned} \tag{4.4.43}$$

This is the theta term for the $O(2m-2)$ NLSM with field $\mathbf{N}$ living in the vortex core, and we see that the theta angle is the same as for the original $O(2m)$ NLSM. This proves our claim from the previous paragraph.

From the discussion above we see that threading a 2π flux of F in the $O(2m)$ NLSM theory will produce m copies of the $O(2m-2)$ theory, since the 2π flux creates a vortex in all m species of bosons, and a vortex in just one species produces one copy of the $O(2m-2)$ NLSM with theta term. We should mention a technical point that the m vortices cannot all be localized at a point and should spread or separate slightly in space after we thread the 2π flux. This is because the amplitude $|b_\ell|$ should vanish at the core of a vortex in the phase of b_ℓ, but the NLSM constraint $\sum_\ell |b_\ell|^2$ does not allow the amplitudes $|b_\ell|$ for all ℓ to simultaneously vanish at a particular point. However, this subtlety does not effect the basic physical point which is that threading the 2π flux of F produces m vortices (at nearly the same point), each of which carries a copy of the lower dimensional BIQH state.

Let us denote the CS level for the response of the $O(2m)$ NLSM with $\theta = 2\pi k$ in $2m-1$ dimensions by N_{2m-1}. From what we have just learned, and from Eq. (4.4.39) for the reduction of the CS term after threading 2π flux, we find that the CS levels for the response of the NLSMs in dimensions $2m-1$ and $2m-3 = 2(m-1)-1$ must obey the recursion relation

$$N_{2m-1} = mN_{2m-3} \ . \tag{4.4.44}$$

We can now iterate this equation to generate

$$N_{2m-1} = (m!)N_1 \ . \tag{4.4.45}$$

This equation gives the electromagnetic response of the $O(2m)$ NLSM with $\theta = 2\pi k$ in terms of the response of the $O(2)$ NLSM in one dimension with $\theta = 2\pi k$. In Appendix C.5 we directly calculate N_1 for the $O(2)$ NLSM (in the limit of large coupling g) and show that $N_1 = -k$ in that case. This then implies that

$$N_{2m-1} = -(m!)k \ , \tag{4.4.46}$$

and this agrees (in magnitude and in sign) with our boundary calculation using gauged WZ actions. Thus, the dimensional reduction approach employed in this subsection gives a clear physical picture for the $m!$ factor in the response, and crucially depends on the fact that all the bosons b_ℓ carry a $U(1)$ charge.

4.5 General Gauge invariance argument for the BIQH response and comparison with the fermionic case

In this section we show that the factor of $m!$ in the BIQH response derived in Sec. 4.4 can be understood by studying large $U(1)$ gauge transformations of the CS action on general (closed, compact) Euclidean manifolds which do not necessarily admit a spin structure. Physically, we require the *exponential* of the CS term to be gauge-invariant, since this object is part of the partition function of a short-range entangled (gapped) phase coupled to the external field A. In such phases, since the ground state is always unique, one can always safely integrate out the matter field and obtain a gauge-invariant action. In contrast, if we do the same thing for a topologically ordered state, for example a Laughlin state, we will indeed get a non-gauge-invariant response theory. This is because the calculation to arrive at a response theory is only perturbatively defined around a single ground state.

The level N_{2m-1} of the CS term must be quantized for the exponential of the CS term to be gauge-invariant, but we find that the required quantization of N_{2m-1} is different depending on whether or not the Euclidean manifold admits a spin structure. Bosonic theories may be formulated on any generic manifold, but the Dirac equation cannot be formulated properly on a manifold which does not admit a spin structure, and so we cannot place fermions on these manifolds. In particular we find that the CS action will be gauge-invariant on a generic manifold if the level N_{2m-1} is quantized in integer multiples of $m!$, which agrees with our direct calculation for the NLSM theory from Sec. 4.4. For the fermionic case we use the Atiyah-Singer index theorem for the twisted Dirac complex[56] to show that the CS response action will not, in general, be $U(1)$ gauge-invariant unless suitable gravitational terms are also included in the response action. We also discuss an explicit example of how these gravitational terms can contribute to the response of a fermionic SPT phase with $U(1)$ symmetry. Furthermore, using these examples, we compare the quantization of FIQH and BIQH states, as well as another type of bosonic SPT state with non-trivial topological electromagnetic-gravitational response.

4.5.1 Gauge invariance argument for bosonic and fermionic states

In Euclidean spacetime the CS term takes the form

$$S_{CS}[A] = -i\frac{N_{2m-1}}{(2\pi)^{m-1}m!}\int_{\mathcal{M}} A\wedge F^{m-1}\ . \tag{4.5.1}$$

Here $\mathcal{M}$ is a $(2m-1)$-dimensional closed, compact manifold, and for the moment let us assume that N_{2m-1} is some number, not necessarily an integer. A more careful way to define the CS term is to consider an extension of the field configuration A into a $2m$-dimensional manifold $\mathcal{B}$ such that $\partial\mathcal{B} = \mathcal{M}$ (this type of analysis of CS terms dates back at least to Ref. [193]). Let $\tilde{A}$ denote this extension. Then the CS term is more properly written as

$$S_{CS}[A] = -i\frac{N_{2m-1}}{(2\pi)^{m-1}m!}\int_{\mathcal{B}} \tilde{F}^m \ , \tag{4.5.2}$$

where $\tilde{F} = d\tilde{A}$. In this formulation, a large $U(1)$ gauge transformation of the action can be understood as a change of the extension of A into the larger space $\mathcal{B}$. Suppose $\tilde{A}^{(1)}$ and $\tilde{A}^{(2)}$ are two different extensions of A. In order for the CS term to be well-defined, we require that the difference

$$-i\frac{N_{2m-1}}{(2\pi)^{m-1}m!}\int_{\mathcal{B}} (\tilde{F}^{(1)})^m - \left(-i\frac{N_{2m-1}}{(2\pi)^{m-1}m!}\int_{\mathcal{B}} (\tilde{F}^{(2)})^m\right) \tag{4.5.3}$$

be an integer multiple of $2\pi i$ so that the exponential of the difference of the two Euclidean actions is equal to one. This is equivalent to the requirement that the exponential of the CS term be invariant under a large $U(1)$ gauge transformation. This difference can in turn be written as the integral of the field strength F of a gauge field in $2m$ dimensions over the closed manifold $2m$-dimensional manifold X constructed by gluing $\mathcal{B}$ to another copy of $\mathcal{B}$ (with the opposite orientation) along their boundary (which is the original lower-dimensional manifold $\mathcal{M}$). So the requirement for a well-defined CS term is to check that

$$I[A] = -i\frac{N_{2m-1}}{(2\pi)^{m-1}m!}\int_{X} F^m \ , \tag{4.5.4}$$

is equal to $2\pi k$ for some integer k, where X is a $2m$-dimensional closed, compact manifold, and F is now the field strength of a gauge field A living in $2m$ dimensions.

We must also make one crucial assumption about the configuration of F on X, which is that F should be chosen to satisfy the Dirac quantization condition

$$\int_{\mathcal{C}} \frac{F}{2\pi} \in \mathbb{Z} \ , \tag{4.5.5}$$

where $\mathcal{C}$ is any non-trivial two-cycle on X (i.e., an element of the second homology group $H_2(X,\mathbb{R})$). This requirement tells us how a general background field F on X can be expanded in terms of the elements of the second cohomology group $H^2(X,\mathbb{R})$ of X (more precisely, we expand F in terms of elements of the second de Rham cohomology group $H^2_{dR}(X)$, which is in turn isomorphic to $H^2(X,\mathbb{R})$ by de Rham's theorem).

If we enforce the Dirac quantization condition of Eq. (4.5.5), then on a generic closed, compact Euclidean manifold

X we have

$$\int_X \left(\frac{F}{2\pi}\right)^m \in \mathbb{Z}\ . \tag{4.5.6}$$

Briefly, this comes from the fact that (assuming the Dirac quantization condition) $\frac{F}{2\pi}$ is the first Chern class c_1 of a complex line bundle over X. The integral over X of its m^{th} power $(c_1)^m$ is then one of the Chern numbers of this complex line bundle, and is therefore an integer[194]. Note that here we also need to assume that X is orientable. From this result we deduce that the (exponential of the) CS term will be invariant under large $U(1)$ gauge transformations on any Euclidean manifold provided that

$$N_{2m-1} = (m!)k\ ,\ k \in \mathbb{Z} \tag{4.5.7}$$

which agrees with our result from Sec. 4.4 derived using the NLSM description of the BIQH state. In Appendix C.2 we show that the minimum value with $\int_X \left(\frac{F}{2\pi}\right)^m = 1$ can be achieved for $X = C\mathbb{P}^m$ if we thread 2π flux of F through the non-trivial two-cycle on $C\mathbb{P}^m$.

We can also compare this result with the result for FIQH phases with $U(1)$ symmetry in the same dimension. In any odd dimension, we can consider the massive Dirac fermion as a model for a FIQH state with the global $U(1)$ symmetry associated to charge conservation. The Lagrangian of a massive Dirac fermion on flat, $(2m-1)$-dimensional Minkowski spacetime takes the form

$$\mathcal{L}_{\text{Dirac}}[\psi, A] = \overline{\psi}(i\not{\partial} - \not{A} - M)\psi\ , \tag{4.5.8}$$

where γ^μ, $\mu = 0, \dots, 2m-2$, are the standard Gamma matrices satisfying $\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}$ with $\eta^{\mu\nu} = \text{diag}(1, -1, -1, ..., -1)$, $\overline{\psi} = \psi^\dagger\gamma^0$, and $M > 0$ is the mass of the Dirac fermion. We also used the Feynman slash notation $\not{\partial} \equiv \gamma^\mu\partial_\mu$, etc. Here we have also coupled the fermion ψ to the background $U(1)$ gauge field (electromagnetic field) A_μ. After integrating out the massive Dirac fermion, we arrive at a topological response theory given by the CS theory at level one:

$$S_{\text{Dirac}}[A] = -i\frac{1}{(2\pi)^{m-1}m!}\int_{\mathcal{M}} A \wedge F^{m-1}\ , \tag{4.5.9}$$

where in this case the spacetime manifold $\mathcal{M}$ is just $(2m-1)$-dimensional Minkowski spacetime. In deriving this response theory we have employed a Pauli-Villars regularization procedure (see Ref. [58] or the more recent discussion in Ref. [70]) such that integrating out a Dirac fermion with a negative mass M does not produce any topological term (i.e., a CS term with level zero). Also, we have omitted all the non-topological terms, for example the Maxwell term, from the final response action. Since a single massive Dirac fermion gives rise to a CS term for A at level one, we have the result that

$$N_{2m-1} \in \mathbb{Z} \tag{4.5.10}$$

for general $U(1)$ fermionic SPT phases in $2m-1$ dimensions.

However, as we know from the discussion of the CS term earlier in this section, on a generic manifold $\mathcal{M}$ the CS term will not be invariant under large $U(1)$ gauge transformations unless the level N_{2m-1} is an integer multiple of $m!$. Thus, one might naively conclude that the response action for the FIQH state on a generic manifold $\mathcal{M}$ is not invariant under large $U(1)$ gauge transformations. Of course, this is not the case. The resolution of this problem is to recall that on a curved manifold $\mathcal{M}$ a Dirac fermion also has non-trivial gravitational and (when coupled to the gauge field A) mixed gauge and gravitational responses. The gravitational part of the response comes from the coupling of the Dirac fermion to the metric $g_{\mu\nu}$ of the curved spacetime $\mathcal{M}$. The response action for the FIQH state (as modeled by the massive Dirac fermion) will include these additional terms. The effective action for a massive Dirac fermion on a $(2m-1)$-dimensional closed, compact manifold $\mathcal{M}$ can be written in the form[60]

$$S_{\text{FIQH}}[A,g] = 2\pi i \int_{\mathcal{B}} \text{ch}(\tilde{F}) \wedge \hat{A}(\mathcal{B}) \ , \tag{4.5.11}$$

where $\partial\mathcal{B} = \mathcal{M}$, $\text{ch}(\tilde{F}) = e^{\frac{\tilde{F}}{2\pi}}$ is the Chern character of the extended field strength $\tilde{F}$ and $\hat{A}(\mathcal{B})$ is the *A-roof genus* (or Dirac genus) on $\mathcal{B}$. Since we are focusing on fermionic phases here, we should only consider spin manifolds $\mathcal{M}$ and $\mathcal{B}$. The A- roof genus $\hat{A}(\mathcal{B})$ can be expressed in terms of the Pontryagin classes $p_i(\mathcal{B})$ of $\mathcal{B}$ as[72],

$$\hat{A}(\mathcal{B}) = 1 - \frac{1}{24}p_1 + \frac{1}{5760}(7p_1^2 - 4p_2) + ..., \tag{4.5.12}$$

with

$$p_1 = -\frac{1}{8\pi^2}\text{Tr}\tilde{\mathcal{R}}^2, \tag{4.5.13}$$

$$p_2 = -\frac{1}{64\pi^4}\text{Tr}\tilde{\mathcal{R}}^4 + \frac{1}{128\pi^4}\left(\text{Tr}\tilde{\mathcal{R}}^2\right)^2. \tag{4.5.14}$$

Here, $\tilde{R}$ is the $2m \times 2m$ matrix of two-forms (curvature two-form) on $\mathcal{B}$:

$$\tilde{R}^{\nu}_{\mu} = \frac{1}{2}\tilde{R}_{\alpha\beta\mu}{}^{\nu} dx^{\alpha} \wedge dx^{\beta} \tag{4.5.15}$$

which depends on the Riemann curvature tensor $\tilde{R}_{\alpha\beta\mu}{}^{\nu}$ in the extended space $\mathcal{B}$. In Eq. (4.5.11) it is understood that the integral is only over the terms of (differential form) degree $2m$ in the product $\text{ch}(\tilde{F}) \wedge \hat{A}(\mathcal{B})$ on $\mathcal{B}$. It is easy to see that when we only consider the electromagnetic response in $S_{\text{FIQH}}[A,g]$ (e.g., by setting all p_i to 0 on $\mathcal{B}$), it recovers the response theory Eq. (4.5.9) of the massive Dirac fermion in $2m-1$ dimensions. More importantly, the response theory $S_{\text{FIQH}}[A,g]$ is fully gauge-invariant. This is because on any closed, compact $2m$-dimensional spin manifold X, the Atiyah-Singer index theorem for the twisted Dirac complex (see, for example, Ref. [56]) states that

$$\int_X \text{ch}(\tilde{F}) \wedge \hat{A}(X) = \text{index}(\not{D}) \in \mathbb{Z}\,, \tag{4.5.16}$$

where index$(\not{D})$ is the index (the difference between the number of positive and negative chirality zero modes) of the Dirac operator on X, and is necessarily an integer. Although we originally derived Eq. (4.5.11) by using the theory of a massive Dirac fermion on the curved manifold $\mathcal{M}$ as a model for the FIQH state, we argue that due to the requirement of large $U(1)$ gauge invariance, Eq. (4.5.11) is the minimal (or "level 1") non-trivial gauge and gravitational response theory of any putative FIQH phase with $U(1)$ symmetry in $(2m-1)$ dimensions.

There is one more subtlety here. When m is even (i.e., when the spacetime dimension is $4k-1$ with $k \in \mathbb{Z}$), the object ch$(\tilde{F}) \wedge \hat{A}(\mathcal{B})$ contains a purely gravitational term that comes from $\hat{A}(\mathcal{B})$ alone. Such a term itself can be well-defined (the index theorem for the untwisted Dirac complex guarantees that it integrates to an integer on a closed, compact spin manifold) and can capture the non-trivial gravitational response of certain short-range entangled states even without the inclusion of a global $U(1)$ symmetry. For example, for $m=2$ the purely gravitational term is given by $-\frac{1}{24}p_1$ on $\mathcal{B}$, which is equivalent to the three-dimensional gravitational Chern-Simons term on $\mathcal{M}$. This term is tied to the chiral central charge. Hence, we can separately consider the purely gravitational term $\hat{A}(\mathcal{B})$ and the rest of the terms $\left[\text{ch}(\tilde{F}) \wedge \hat{A}(\mathcal{B}) - \hat{A}(\mathcal{B})\right]$ in Eq. (4.5.11).

In general, we can consider the FIQH phase at level $N_{2m-1} \in \mathbb{Z}$, whose topological response theory (minus the purely gravitational term) is given by

$$S'_{\text{FIQH}}[A,g] = 2\pi i N_{2m-1} \int_{\mathcal{B}} \left[\text{ch}(\tilde{F}) \wedge \hat{A}(\mathcal{B}) - \hat{A}(\mathcal{B})\right]\,. \tag{4.5.17}$$

$S'_{\text{FIQH}}[A,g]$ naturally contains both a term capturing the electromagnetic response of the FIQH state and other terms that describe various different types of mixed gauge-gravitational response. The coexistence of all these terms is enforced by the properties of spin manifolds and the Atiyah-Singer index theorem, and reflects the fermionic nature of the FIQH phase. This combination also informs us that we should *not* use each of the terms to independently classify fermionic SPTs with $U(1)$ symmetry. For bosonic systems, we can, in principle, separately study each single term in $S'_{\text{FIQH}}[A,g]$ by itself, and use each of them to characterize a different class of bosonic SPTs. However, just like the quantization of the level of the $U(1)$ CS term, we expect gauge invariance to enforce a larger quantization unit of the "level" when we isolate a single term as a bosonic response theory, as opposed to the case where that term appears in the full combination $S'_{\text{FIQH}}[A,g]$ as a part of a fermionic theory. The difference in the quantization unit of the "level" between fermionic and bosonic systems will also lead to very different behaviors under dimensional reduction, the details of which will be elaborated using examples. In Sec. 4.5.2, we provide an example of the electromagnetic and gravitational response theory of FIQH states in five dimensions. In Sec. 4.5.3, we compare the example fermionic response theory with five-dimensional bosonic theories, including the BIQH state and another type of bosonic $U(1)$ SPT state with non-trivial mixed electromagnetic and gravitational response.

4.5.2 An example of electromagnetic and gravitational response theories of FIQH states and their dimensional reduction

In this section, we restrict our discussion to the topological response theory of a five-dimensional FIQH phase, and we study its dimensional reduction to the response theory of a FIQH state in three dimensions. We start with the response theory of the FIQH phase at level $N_5 = 1$ on a five-dimensional spin manifold $\mathcal{M}^5$:

$$S_{\text{FIQH}}[A,g] = 2\pi i \int_{\mathcal{B}^6} \left[\frac{1}{6}\left(\frac{\tilde{F}}{2\pi}\right)^3 - \frac{p_1}{24} \wedge \frac{\tilde{F}}{2\pi} \right] , \tag{4.5.18}$$

where $\mathcal{B}^6$ is a six-dimensional spin manifold such that $\mathcal{M}^5 = \partial\mathcal{B}^6$. We first consider its dimensional reduction to the response theory of a FIQH state in three dimensions. In order to do so, we take the spacetime manifold to be $\mathcal{M}^5 = S^2 \times \mathcal{M}^3$ where $\mathcal{M}^3$ is a closed, compact three-dimensional manifold, and S^2 is a two-sphere. In this case, it is natural to consider the bounding space $\mathcal{B}^6 = S^2 \times \mathcal{B}^4$ where $\mathcal{B}^4$ is a four-dimensional spin manifold such that $\mathcal{M}^3 = \partial\mathcal{B}^4$. Also, we consider the configuration with 2π flux of $\tilde{F}$ piercing the S^2 part. The response theory is then reduced to

$$\begin{aligned} S_{\text{FIQH}}[A,g]\Big|_{S^2\times\mathcal{M}^3} &= 2\pi i \int_{\mathcal{B}^4} \left(\frac{\tilde{F}^2}{8\pi^2} - \frac{p_1}{24} \right) \\ &= i \int_{\mathcal{M}^3} \left[\frac{A \wedge F}{4\pi} - \frac{1}{24}\frac{1}{4\pi}\text{Tr}\left(\omega \wedge d\omega + \frac{2}{3}\omega\wedge\omega\wedge\omega \right) \right] , \end{aligned} \tag{4.5.19}$$

where ω is the $SO(1,2)$ spin connection on $\mathcal{M}^3$. The first term describes the standard Integer Quantum Hall effect in three dimensions with unit Hall conductance. The second term, which is the gravitational Chern-Simons term, captures the gravitational response of a three-dimensional chiral state with chiral central charge $c = 1$. On the other hand, we can directly consider a five-dimensional massive Dirac fermion as a model of a five-dimensional FIQH state at level one on this background. When put on the manifold $S^2 \times \mathcal{M}^3$ with 2π flux of F inside the S^2 part, the five-dimensional massive Dirac fermion effectively reduces to a three-dimensional massive Dirac fermion on $\mathcal{M}^3$ at low energies when the linear size of the S^2 part is small compared to the length scale set by the Dirac fermion mass M. The $U(1)$ and gravitational response of the three-dimensional FIQH state is indeed given by the dimensionally-reduced response theory $S_{\text{FIQH}}[A, g]\Big|_{S^2\times\mathcal{M}^3}$.

Finally, let us also remark here that the response theory Eq. (4.5.18) for the five-dimensional FIQH state can also be used to derive the electromagnetic and gravitational responses of a topological superconductor in four dimensions using a dimensional reduction procedure[195].

4.5.3 Comparing bosonic and fermionic systems: quantization and dimensional reduction

As we have discussed, we can consider each term of $S_{\text{FIQH}}[A, g]$ separately as a topological response theory for bosonic $U(1)$ SPTs in five dimensions:

$$S_{\text{BIQH}}[A] = 2\pi i N_5 \int_{\mathcal{B}^6} \frac{1}{6}\left(\frac{\tilde{F}}{2\pi}\right)^3 , \tag{4.5.20}$$

$$S_{\text{BSPT}}[A, g] = -2\pi i N_5' \int_{\mathcal{B}^6} \frac{p_1}{24} \wedge \frac{\tilde{F}}{2\pi} . \tag{4.5.21}$$

$S_{\text{BIQH}}[A]$ is the response theory of a five-dimensional BIQH state, and requires a quantization of level as $N_5 \in 6\mathbb{Z}$ as we showed in this section and in Sec. 4.4. $S_{\text{BSPT}}[A, g]$ characterizes an independent class of bosonic SPT states in five dimensions without a requirement of $U(1)$ symmetry[196]. Similar to the BIQH and FIQH cases, gauge invariance requires $N_5' \int_{X^6} \frac{p_1}{24} \wedge \frac{\tilde{F}}{2\pi} \in \mathbb{Z}$ on any closed six-dimensional manifold X^6. Since p_1 and $\frac{\tilde{F}}{2\pi}$ are both cohomology classes of X^6 with integer coefficients, gauge invariance then enforces the quantization $N_5' \in 24\mathbb{Z}$. We would like to point out that previously Ref. [196] considered only closed six-dimensional manifolds that can be decomposed into products of two and four-dimensional manifolds, and concluded that $N_5' \in 8\mathbb{Z}$. However, when we take into account more general six-dimensional manifolds, for example $C\mathbb{P}^3$, we arrive at the stronger quantization condition $N_5' \in 24\mathbb{Z}$. [3] As seen here, for both of the bosonic theories $S_{\text{BIQH}}[A]$ and $S_{\text{BSPT}}[A, g]$, the quantization units of their levels are larger than when these two terms appear together in the fermionic theory $S_{\text{FIQH}}[A, g]$ in Eq. (4.5.18).

Now let us consider a similar dimensional reduction of both $S_{\text{BIQH}}[A_\mu]$ and $S_{\text{BSPT}}[A_\mu, g]$ to three dimensions, as we did in the fermion case. Now the five-dimensional spacetime manifold $\mathcal{M}^5$ is taken to be the product $S^2 \times \mathcal{M}^3$ with $\mathcal{M}^3$ a three-dimensional manifold. Again, we consider the configuration with 2π flux of $\tilde{F}$ piercing the S^2 part.

[3] When we consider $C\mathbb{P}^3$ with the $U(1)$ gauge field given by its fundamental line bundle, we find that $N_5' \int_{C\mathbb{P}^3} \frac{p_1}{24} \wedge \frac{\tilde{F}}{2\pi} = N_5'/6$. Combining with the result $N_5' \in 8\mathbb{Z}$ from Ref. [196], we can conclude that the gauge invariance argument requires $N_5' \in 24\mathbb{Z}$. On the other hand, since p_1 and $\frac{\tilde{F}}{2\pi}$ are both cohomology classes with integer coefficients, any $N_5' \in 24\mathbb{Z}$ will satisfy the gauge invariance requirement.

The dimensionally reduced theories are given by

$$S_{\text{BIQH}}[A]\Big|_{S^2\times\mathcal{M}^3} = i2\pi\frac{N_5}{2}\int_{\mathcal{M}^3}\frac{A\wedge F}{(2\pi)^2},$$
$$S_{\text{BSPT}}[A,g]\Big|_{S^2\times\mathcal{M}^3} = -i2\pi\frac{N_5'}{24}\int_{\mathcal{M}^3}\frac{1}{4\pi}\text{Tr}\left(\omega\wedge d\omega+\frac{2}{3}\omega\wedge\omega\wedge\omega\right). \tag{4.5.22}$$

For the BIQH state, due to the bosonic quantization $N_5 \in 6\mathbb{Z}$, we notice that the most fundamental three-dimensional BIQH state (with CS level $N_3 = 2$) cannot be realized from such a dimensional reduction from a five-dimensional BIQH state. From our analysis of the CS level of the BIQH state, it should be generally true that there are certain lower-dimensional BIQH states that cannot be realized from the dimensional reduction of higher-dimensional BIQH states. In fact, this phenomenon is not restricted to BIQH states. For the bosonic SPT states described by Eq. (4.5.21), due to the quantization $N_5' \in 24\mathbb{Z}$, the action $S_{\text{BSPT}}[A,g]\Big|_{S^2\times\mathcal{M}^3}$ only captures chiral bosonic states with chiral central charge $c \in 24\mathbb{Z}$. The E_8 state in $(2+1)$ dimensions, which has chiral central charge $c = 8$, is absent in this dimensional reduction picture. This is in strong contrast with the fermionic theory studied in Sec. 4.5.2, in which case lower-dimensional response theories of FIQH at any level can be obtained from dimensionally reducing higher-dimensional FIQH states.

4.6 Electromagnetic response of BTI states in all even dimensions

In this section we construct the gauged WZ action for the boundary of BTI states in all even dimensions. Again, the action that we construct satisfies the gauging principle of Eq. (4.3.17). Unlike the BIQH case, however, the gauged boundary action that we find for BTI states *is* completely gauge-invariant. From the form of the gauged action for the boundary of the BTI, we find that if the NLSM field on the boundary condenses in such a way that the $\mathbb{Z}_2$ symmetry of the BTI is broken, but the $U(1)$ symmetry remains intact, then the boundary of the BTI can exhibit a $\mathbb{Z}_2$ symmetry-breaking Quantum Hall response (recall from Sec. 4.3 that the BTI phase also has a $\mathbb{Z}_2$ symmetry such that the total symmetry group is $U(1)\rtimes\mathbb{Z}_2$) [4]. We find that the boundary Quantum Hall response is characterized by a CS level N_{2m-1} which is quantized in units of $\frac{m!}{2}$, i.e., the minimal boundary Quantum Hall response is half that of the minimal BIQH state that can be realized intrinsically in the same spacetime dimension. This boundary response implies a bulk response of the form of Eq. (4.2.2) with the parameter Θ_{2m} quantized as $\Theta_{2m} = 2\pi\left(\frac{m!}{2}\right)$.

In Appendix C.1 we re-interpret the gauged action constructed in this section in terms of $U(1)$-equivariant cohomology of the sphere S^{2m}. There we show that the problem of constructing a gauged WZ action for the boundary of

[4] Our result can also be applied to systems with a symmetry of the form $U(1)\times\mathbb{Z}_2$, but only in the case that the $U(1)$ symmetry rotates the maximal number of components of n_a as in the $U(1)\rtimes\mathbb{Z}_2$ cases considered in this Chapter. For example, according to Ref. [35] bosonic SPT phases in four dimensions with $U(1)\times\mathbb{Z}_2^T$ symmetry have a $(\mathbb{Z}_2)^3$ classification. However, only one of the three root phases is described by an $O(5)$ NLSM with symmetry assignment that rotates four out of the five components of $\mathbf{n}$[19], so this is the only case in which our technique can be applied directly. For the other cases one must use the more general methods of Ref. [41] to gauge the $U(1)$ symmetry.

the BTI phase in $2m$ dimensions is equivalent to the problem of constructing an equivariant extension of ω_{2m}, the volume form for S^{2m}, and we explicitly construct such an extension. The fact that an extension exists is mathematically equivalent to the result in this section that the gauged WZ action for the boundary of the BTI is completely gauge-invariant. We also show that the forms $\Phi^{(r)}$ which appear later in this section in the counterterms of Eq. (4.6.25) are exactly the same forms which are needed for the construction of the equivariant extension of ω_{2m}.

We now construct the gauged WZ action for the boundary of BTI states. Recall that in the BTI case we define the integer m via $2m+1 = d+2$, so that the SPT phases we study live in $2m$ spacetime dimensions and have a $2m-1$ dimensional boundary (the bulk spacetime dimension was defined to be $d+1$). We again make use of the forms $\mathcal{J}_\ell$ and $\mathcal{K}_\ell$, $\ell = 1, \dots, m$ defined in Eqs. (4.4.2). Now, however, the NLSM field has the extra component n_{2m+1}, so the relations of Eq. (4.4.16) are replaced with

$$\sum_{\ell=1}^{m}(n_{2\ell-1}^2 + n_{2\ell}^2) = 1 - n_{2m+1}^2 \tag{4.6.1a}$$

$$\sum_{\ell=1}^{m}(n_{2\ell-1}dn_{2\ell-1} + n_{2\ell}dn_{2\ell}) = -n_{2m+1}dn_{2m+1} \ . \tag{4.6.1b}$$

In this case the WZ term takes the form

$$S_{WZ}[\mathbf{n}] = \frac{2\pi k}{\mathcal{A}_{2m}}\int_{\mathcal{B}} \omega_{2m} \ , \tag{4.6.2}$$

where $\mathcal{B} = [0,1] \times \mathbb{R}^{d-1,1}$ is the extended boundary spacetime.

For the BTI case it is convenient to define the forms $\Phi^{(r)}$ for $r = 0, 1, \dots, m-1$ as

$$\Phi^{(r)} = \sum_{\ell_1,\dots,\ell_{m-r}=1}^{m} \mathcal{K}_{\ell_1}\wedge\cdots\wedge\mathcal{K}_{\ell_{m-r}} \ , \tag{4.6.3}$$

and in addition we define $\Phi^{(m)} = 1$, so that $\Phi^{(r)}$ is defined for all $r = 0, 1, \dots, m$. Also, note that all of these forms are closed since each $\mathcal{K}_\ell$ is closed. Just as in the BIQH case, the essential ingredient in the construction of the gauged WZ action is a formula for how these forms change under a gauge transformation.

Claim: Under a gauge transformation $b_\ell \to e^{i\xi}b_\ell$ we have $\Phi^{(r)} \to \Phi^{(r)} + \delta_\xi\Phi^{(r)}$ with

$$\delta_\xi\Phi^{(r)} = -(m-r)n_{2m+1}dn_{2m+1}\wedge\Phi^{(r+1)}\wedge d\xi \ . \tag{4.6.4}$$

Proof: Using the symmetry of the summand of $\Phi^{(r)}$ under the exchange of any two of the indices $\ell_1, \dots, \ell_{m-r}$, we first find that

$$\delta_\xi \Phi^{(r)} = (m-r) \sum_{\ell_1,\ldots,\ell_{m-r}=1}^{m} (n_{2\ell_1-1} dn_{2\ell_1-1} + n_{2\ell_1} dn_{2\ell_1}) \wedge d\xi \wedge \mathcal{K}_{\ell_2} \wedge \cdots \wedge \mathcal{K}_{\ell_{m-r}} \ . \tag{4.6.5}$$

Now we can move $d\xi$ all the way to the right by commuting it past the two-forms $K_{\ell_2}, \ldots, K_{\ell_{m-r}}$. This gives

$$\delta_\xi \Phi^{(r)} = (m-r) \sum_{\ell_1=1}^{m} (n_{2\ell_1-1} dn_{2\ell_1-1} + n_{2\ell_1} dn_{2\ell_1}) \wedge \Phi^{(r+1)} \wedge d\xi \ , \tag{4.6.6}$$

where we used the fact that

$$\Phi^{(r+1)} = \sum_{\ell_2,\ldots,\ell_{m-r}=1}^{m} \mathcal{K}_{\ell_2} \wedge \cdots \wedge \mathcal{K}_{\ell_{m-r}} \ . \tag{4.6.7}$$

Finally we can do the sum over ℓ_1 using the second relation of Eqs. (4.6.1), and this gives the final formula of Eq. (4.6.4). ■

In terms of the form $\Phi^{(0)}$ we can write the volume form on S^{2m} as

$$\omega_{2m} = \frac{1}{(m-1)!} \left[\sum_{\ell_1,\ldots,\ell_m=1}^{m} \mathcal{J}_{\ell_1} \wedge \mathcal{K}_{\ell_2} \wedge \cdots \wedge \mathcal{K}_{\ell_m} \wedge dn_{2m+1} + \frac{n_{2m+1}}{m} \Phi^{(0)} \right] . \tag{4.6.8}$$

The last term in this expression is just the term

$$n_{2m+1} dn_1 \wedge dn_2 \wedge \cdots \wedge dn_{2m-1} \wedge dn_{2m} \ , \tag{4.6.9}$$

but re-written using the formula

$$dn_1 \wedge dn_2 \wedge \cdots \wedge dn_{2m-1} \wedge dn_{2m} = \frac{1}{m!} \sum_{\ell_1,\ldots,\ell_m=1}^{m} \mathcal{K}_{\ell_1} \wedge \cdots \wedge \mathcal{K}_{\ell_m} \ . \tag{4.6.10}$$

We are now in a position to construct the properly gauged action step by step as in Section 4.4 on the BIQH system. We demonstrate the first few steps in the construction and then write down the final answer. To start we have

$$\begin{aligned} \delta_\xi \omega_{2m} &= -\frac{1}{(m-1)!} dn_{2m+1} \wedge \Phi^{(1)} \wedge d\xi \\ &= -\frac{1}{(m-1)!} d\left(n_{2m+1} \Phi^{(1)} \wedge d\xi \right) \ . \end{aligned} \tag{4.6.11}$$

This is computed using Eq. (4.6.4) for the case $r = 0$ combined with the formula

$$\delta_\xi \left(\sum_{\ell_1,\dots,\ell_m=1}^{m} \mathcal{J}_{\ell_1} \wedge \mathcal{K}_{\ell_2} \wedge \cdots \wedge \mathcal{K}_{\ell_m} \wedge dn_{2m+1} \right) = -(1 - n_{2m+1}^2) dn_{2m+1} \wedge \Phi^{(1)} \wedge d\xi \ , \tag{4.6.12}$$

which is easily proven using Eq. (4.4.3) and Eq. (4.6.1). Then we have

$$\delta_\xi S_{WZ}[\mathbf{n}] = -\frac{2\pi k}{\mathcal{A}_{2m}} \frac{1}{(m-1)!} \int_{\mathbb{R}^{d-1,1}} n_{2m+1} \Phi^{(1)} \wedge d\xi \ . \tag{4.6.13}$$

We therefore choose the first counterterm to be

$$S_{ct}^{(1)}[\mathbf{n}, A] = \frac{2\pi k}{\mathcal{A}_{2m}} \frac{1}{(m-1)!} \int_{\mathbb{R}^{d-1,1}} n_{2m+1} \Phi^{(1)} \wedge A \ . \tag{4.6.14}$$

The total gauged WZ action is now

$$S'_{gauged,WZ}[\mathbf{n}, A] = S_{WZ}[\mathbf{n}] + S_{ct}^{(1)}[\mathbf{n}, A] \ , \tag{4.6.15}$$

and under a gauge transformation we find

$$\delta_\xi S'_{gauged,WZ}[\mathbf{n}, A] = -\frac{2\pi k}{\mathcal{A}_{2m}} \frac{1}{(m-2)!} \int_{\mathbb{R}^{d-1,1}} n_{2m+1}^2 dn_{2m+1} \wedge \Phi^{(2)} \wedge d\xi \wedge A \ . \tag{4.6.16}$$

Next we integrate by parts using the formula

$$d\left(\frac{1}{3} n_{2m+1}^3 \Phi^{(2)} \wedge d\xi \wedge A \right) = n_{2m+1}^2 dn_{2m+1} \wedge \Phi^{(2)} \wedge d\xi \wedge A - \frac{1}{3} n_{2m+1}^3 \Phi^{(2)} \wedge d\xi \wedge F \ , \tag{4.6.17}$$

to find (neglecting boundary terms)

$$\delta_\xi S'_{gauged,WZ}[\mathbf{n}, A] = -\frac{2\pi k}{\mathcal{A}_{2m}} \frac{1}{(m-2)!} \frac{1}{3} \int_{\mathbb{R}^{d-1,1}} n_{2m+1}^3 \Phi^{(2)} \wedge d\xi \wedge F \ . \tag{4.6.18}$$

We should then take the second counterterm to be

$$S_{ct}^{(2)}[\mathbf{n}, A] = \frac{2\pi k}{\mathcal{A}_{2m}} \frac{1}{(m-2)!} \frac{1}{3} \int_{\mathbb{R}^{d-1,1}} n_{2m+1}^3 \Phi^{(2)} \wedge A \wedge F \ . \tag{4.6.19}$$

To see the full structure of the counterterms it is necessary to go one step further. At this point the total gauged action is

$$S''_{gauged,WZ}[\mathbf{n},A] = S_{WZ}[\mathbf{n}] + S^{(1)}_{ct}[\mathbf{n},A] + S^{(2)}_{ct}[\mathbf{n},A]\ , \tag{4.6.20}$$

and under a gauge transformation we have

$$\delta_\xi S''_{gauged,WZ}[\mathbf{n},A] = -\frac{2\pi k}{\mathcal{A}_{2m}}\frac{1}{(m-3)!}\frac{1}{3}\int_{\mathbb{R}^{d-1,1}} n^4_{2m+1} dn_{2m+1}\wedge\Phi^{(3)}\wedge d\xi\wedge A\wedge F\ . \tag{4.6.21}$$

We again integrate by parts to find that

$$\delta_\xi S''_{gauged,WZ}[\mathbf{n},A] = -\frac{2\pi k}{\mathcal{A}_{2m}}\frac{1}{(m-3)!}\frac{1}{5\cdot 3}\int_{\mathbb{R}^{d-1,1}} n^5_{2m+1}\Phi^{(3)}\wedge d\xi\wedge F^2\ . \tag{4.6.22}$$

Note that the denominator contains the *double factorial* $5!! = 5\cdot 3 = 5\cdot 3\cdot 1$. In general, we find that all of the counterterms contain a double factorial. Then the third counterterm takes the form

$$S^{(3)}_{ct}[\mathbf{n},A] = \frac{2\pi k}{\mathcal{A}_{2m}}\frac{1}{(m-3)!}\frac{1}{5!!}\int_{\mathbb{R}^{d-1,1}} n^5_{2m+1}\Phi^{(3)}\wedge A\wedge F^2\ . \tag{4.6.23}$$

At this point the pattern is clear. Continuing with this procedure we find that a total of m counterterms are needed to construct a gauged boundary action which satisfies Eq. (4.3.17), and the final gauged action is *completely gauge-invariant*. It takes the form

$$S_{WZ,gauged}[\mathbf{n},A] = S_{WZ}[\mathbf{n}] + \sum_{r=1}^{m} S^{(r)}_{ct}[\mathbf{n},A]\ , \tag{4.6.24}$$

where the r^{th} counterterm is

$$S^{(r)}_{ct}[\mathbf{n},A] = \frac{2\pi k}{\mathcal{A}_{2m}}\frac{1}{(m-r)!}\frac{1}{(2r-1)!!}\int_{\mathbb{R}^{d-1,1}} (n_{2m+1})^{2r-1}\ \Phi^{(r)}\wedge A\wedge F^{r-1}\ , \tag{4.6.25}$$

where $(2r-1)!!$ is the double factorial,

$$(2r-1)!! = (2r-1)(2r-3)\cdots(3)(1)\ . \tag{4.6.26}$$

The final counterterm is just

$$S^{(m)}_{ct}[\mathbf{n},A] = \frac{2\pi k}{\mathcal{A}_{2m}}\frac{1}{(2m-1)!!}\int_{\mathbb{R}^{d-1,1}} (n_{2m+1})^{2m-1}\ A\wedge F^{m-1}\ , \tag{4.6.27}$$

and its change under a gauge transformation comes only from the transformation of A (the last component n_{2m+1} of the NLSM field does not transform under $U(1)$). This explains why the final gauged action is completely gauge-invariant: the change due to the transformation of A in the last term cancels the transformation from the previous counterterm in the action, and there are no further changes in the last term which remain to be canceled.

Now let us show that the boundary of a BTI phase exhibits a $\mathbb{Z}_2$ symmetry breaking response when the field n_a condenses in such a way that it preserves the $U(1)$ symmetry, but breaks the $\mathbb{Z}_2$ symmetry. The only possible way for n_a to condense and fulfill these requirements is to have

$$n_{2m+1} = \pm 1, \tag{4.6.28a}$$

$$n_a = 0,\ \forall\, a \neq 2m+1\ . \tag{4.6.28b}$$

In this case, all terms in $S_{WZ,gauged}[\mathbf{n}, A]$ vanish except for the final counterterm ($r = m$), which gives the boundary electromagnetic response,

$$S_{eff,bdy}[A] = \pm \frac{2\pi k}{\mathcal{A}_{2m}} \frac{1}{(2m-1)!!} \int_{\mathbb{R}^{d-1,1}} A \wedge F^{m-1}\ , \tag{4.6.29}$$

where we used $0! = 1$ and $\Phi^{(m)} = 1$. Now we use the formulas

$$\mathcal{A}_{2m} = \frac{2\pi^m \sqrt{\pi}}{\Gamma(m + \frac{1}{2})}\ , \tag{4.6.30}$$

and

$$(2m-1)!! = \frac{2^m}{\sqrt{\pi}} \Gamma(m + \tfrac{1}{2})\ , \tag{4.6.31}$$

to find

$$S_{eff,bdy}[A] = \pm \frac{1}{2} \frac{k}{(2\pi)^{m-1}} \int_{\mathbb{R}^{d-1,1}} A \wedge F^{m-1}\ . \tag{4.6.32}$$

Comparing to Eq. (4.2.1), we see that this is a CS response with level

$$N_{2m-1} = \pm \left(\frac{m!}{2} \right) k\ , \tag{4.6.33}$$

which is exactly *half* the response of the BIQH state which appears intrinsically in the same spacetime dimension (which we calculated in Section 4.4). As we discussed in Sec. 4.3, this boundary CS response is equivalent to a bulk electromagnetic response of the form of Eq. (4.2.2) with response parameter

$$\Theta_{2m} = 2\pi \left(\frac{m!}{2} \right) k\ . \tag{4.6.34}$$

However, we should recall from the discussion in Sec. 4.3 that the BTI phase with $k = 2$ is smoothly connected to the phase with $k = 0$. More generally the BTI phase with $\theta = 2\pi k$ is smoothly connected to the phase with $\theta = 2\pi(k \pm 2)$.

This means that the single non-trivial BTI phase is represented by the choice $k = 1$.

Finally, we note that the boundary of the BTI can be driven into the $\mathbb{Z}_2$ symmetry breaking phase without explicitly breaking the $\mathbb{Z}_2$ symmetry. This can be done by adding a term $\mu\, n_{2m+1}^2$ to the Lagrangian. This term is invariant under the full $U(1) \rtimes \mathbb{Z}_2$ symmetry of the BTI but, for $\mu > 0$ and sufficiently large, will drive the system into a phase in which the $\mathbb{Z}_2$ symmetry is spontaneously broken and $n_a = \pm\delta_{a,2m+1}$ (i.e., $n_{2m+1} = \pm 1$ and $n_a = 0$ for $a \neq 2m+1$).

4.7 Applications

In this section we explore several applications of the results obtained so far. We start with the observation that the gauged boundary action for the BIQH state in $2m-1$ spacetime dimensions can be used as building block to construct a bosonic analogue of a Weyl, or chiral, semi-metal in *any* even dimension. We refer to this state as a bosonic chiral semi-metal (BCSM). We write down an effective theory for this state in any even dimension d, compute its electromagnetic response, and compare this response with the response of an ordinary fermionic chiral semi-metal. This construction represents a generalization to higher even dimensions of the work in Ref. [34] that constructed a bosonic analogue of a *Dirac* semi-metal in three dimensions.

As a second application, we show that the boundary theory of the BTI exhibits a bosonic analogue of the parity anomaly of a single Dirac fermion in odd dimensions. As we discuss below, this is closely related to the fact (derived in Sec. 4.6) that the boundary theory of the BTI can exhibit a half-quantized BIQH state when the $\mathbb{Z}_2$ symmetry of the BTI is broken *spontaneously* at the boundary. This situation is clearly analogous to the time-reversal symmetry-breaking half-quantized Integer Quantum Hall state which appears on the surface of the familiar electron topological insulator[139]. This leads us to argue that the boundary theory for a BTI state in $2m$ dimensions cannot exist intrinsically in $2m-1$ dimensions without breaking (partially or fully) the symmetry of the BTI state.

Finally, we perform a detailed study of $\mathbb{Z}_2$ symmetry-breaking domain walls on the boundary of BTI states. We use a dimensional reduction formula for NLSMs with WZ term, analogous to the dimensional reduction formula for theta terms that we derive in Appendix C.4, to show that a $\mathbb{Z}_2$ symmetry-breaking domain wall on the boundary of a BTI state in $2m$ dimensions hosts a lower-dimensional theory which is identical to the boundary theory of the BIQH state in $2m-1$ dimensions. We show that the $U(1)$ anomaly of the theory on the domain wall is exactly canceled by an inflow of charge from the two $\mathbb{Z}_2$ breaking regions on either side of the domain wall. This calculation is an important consistency check for our results on the response of BIQH and BTI states, and also provides a clear example of the phenomenon of anomaly inflow in the context of bosonic SPT phases.

4.7.1 Bosonic analogue of a Weyl semi-metal in any even dimension

In this section we describe how a bosonic analogue of a Weyl semi-metal can be constructed in any even spacetime dimension d using two copies of an $O(d+2)$ NLSM with Wess-Zumino (WZ) term. Before discussing the bosonic analogue, let us first review the basic construction of a Weyl (or more generally a chiral) semi-metal of fermions in any even dimension d. Note that our construction here still assumes a point-like structure of the Fermi surface even in higher dimensions, as opposed to the recent construction in Ref. [197] using Weyl sheets in six spacetime dimensions. We consider a Dirac fermion Ψ in d dimensions. To write down an action for a Dirac fermion we need the gamma matrices γ^μ, $\mu = 0, \dots, d-1$, which obey the Clifford algebra $\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}$ (and we choose $\eta = \text{diag}(1, -1, \dots, -1)$). When d is even we have an extra element $\overline{\gamma}$ of the Clifford algebra which anti-commutes with the other gamma matrices and can be chosen to satisfy $\overline{\gamma}^\dagger = \overline{\gamma}$ and $\overline{\gamma}^2 = \mathbb{I}$ ($\overline{\gamma}$ is the higher-dimensional analog of γ^5 in $d = 4$). In the basis (known as the Weyl basis in $d = 4$) in which $\overline{\gamma}$ takes the block diagonal form

$$\overline{\gamma} = \begin{pmatrix} \mathbb{I} & 0 \\ 0 & -\mathbb{I} \end{pmatrix} , \tag{4.7.1}$$

the fermion Ψ breaks up into chiral and anti-chiral parts as

$$\Psi = (\Psi_+, \Psi_-)^T . \tag{4.7.2}$$

Now a minimal, two-node chiral (or Weyl) semi-metal (CSM) in d dimensions is described at low energies by chiral fermions $\Psi_\pm$ separated in momentum by $2\mathbf{B}$ and in energy by $2B_t$, where $\mathbf{B} = (B_1, \dots, B_{d-1})$ should be thought of as a vector in a $(d-1)$-dimensional momentum space (or Brillouin zone). We assume here that the components B_μ ($\mu = 0, \dots, d-1$, $B_0 = B_t$) are constant, although the results below are expected to hold approximately if the components B_μ are slowly varying functions of x^μ. In addition, both chiral fermions carry charge e of an external $U(1)$ gauge field A_μ. Using the extra gamma matrix $\overline{\gamma}$, an action capturing this low-energy physics takes the form

$$S_{CSM}[\Psi, A, B] = \int d^d x \, i\overline{\Psi}(\partial\!\!\!/ - ieA\!\!\!/ - iB\!\!\!/\overline{\gamma})\Psi , \tag{4.7.3}$$

where $\overline{\Psi} = \Psi^\dagger\gamma^0$ and we used the Feynman slash notation $\partial\!\!\!/ = \gamma^\mu\partial_\mu$, etc. In addition, we have assumed that the separation of $\Psi_\pm$ in momentum and energy is symmetric about $B_\mu = 0$, so that $\Psi_\pm$ is located at $\pm B_\mu$ in momentum/energy space. We also note here that in this low-energy description, the chiral fermion fields $\Psi_\pm$ couple only to the linear combinations $eA_\mu \pm B_\mu$ of the vector fields A_μ and B_μ. This feature will be important later in our construction of a bosonic analogue of the CSM.

The quasi-topological part of the electromagnetic response of the CSM follows directly from the *axial anomaly* of a Dirac fermion in d dimensions[32]. This is because this response is generated by attempting to remove the coupling to B_μ from the action via the chiral rotation

$$\Psi \to e^{i\xi\overline{\gamma}}\Psi \ , \tag{4.7.4}$$

with the parameter ξ chosen as

$$\xi = B_\mu x^\mu \ . \tag{4.7.5}$$

This transformation removes the coupling to B_μ from the action. The physical interpretation of this transformation is that it moves the two cones of the chiral semi-metal to the origin of the Brillouin zone. However, the path integral measure is not invariant under this transformation. Instead, the change in the path integral measure generates a new term in the action of the form ("f" stands for fermionic)

$$S^{(f)}_{eff}[A,B] = -\frac{2}{(\frac{d}{2})!}\left(\frac{e}{2\pi}\right)^{\frac{d}{2}}\int_{\mathbb{R}^{d-1,1}} \xi\, (F)^{\frac{d}{2}} \ . \tag{4.7.6}$$

Noting that $d\xi = B_\mu dx^\mu \equiv B$ (for constant B_μ), and integrating by parts gives the final form of the chiral semi-metal response

$$S^{(f)}_{eff}[A,B] = \frac{2}{(\frac{d}{2})!}\left(\frac{e}{2\pi}\right)^{\frac{d}{2}}\int_{\mathbb{R}^{d-1,1}} B\wedge A\wedge (F)^{\frac{d}{2}-1} \ . \tag{4.7.7}$$

It is also interesting to consider the form Eq. (4.7.6) of the semi-metal response (before integrating by parts), as it has the form of the "Chern character" terms discussed earlier in the Chapter, but with a spacetime-dependent angle $\xi = B_\mu x^\mu$ appearing in the integrand.

So under the chiral transformation of Eq. (4.7.4), the CSM action of Eq. (4.7.3) transforms as

$$S_{CSM}[\Psi,A,B] \to S_{CSM}[\Psi,A,0] + S^{(f)}_{eff}[A,B] \ , \tag{4.7.8}$$

where we again emphasize that the term $S^{(f)}_{eff}[A,B]$ was generated by the change in the path integral measure under the chiral transformation of Eq. (4.7.4). Thus, we can say that the electromagnetic response of the CSM with non-zero separation vector B_μ differs from the response of a CSM with separation vector $B_\mu = 0$ (i.e., a system where the two chiral parts of the Dirac fermion sit at the same point in momentum space) by the term $S^{(f)}_{eff}[A,B]$ from Eq. (4.7.7). For $d = 2$ and $d = 4$ the responses are

$$S^{(f)}_{eff}[A,B] = \frac{e}{\pi}\int_{\mathbb{R}^{1,1}} B\wedge A \ , \tag{4.7.9}$$

and

$$S_{eff}^{(f)}[A,B] = \frac{e^2}{4\pi^2}\int_{\mathbb{R}^{3,1}} B \wedge A \wedge F \ , \tag{4.7.10}$$

respectively. We see that the general expression of Eq. (4.7.7) agrees with the known expressions in low dimensions [32, 33].

Having reviewed the basic properties of fermionic chiral semi-metals, we are now ready to describe our construction of a bosonic analogue of a CSM (BCSM). To motivate our construction we note that the low-energy theory of the CSM has (at least) two essential properties which we use as a guide to construct the BCSM model. The first property is that the CSM model is constructed from two building blocks, namely the chiral fermion theories with fields $\Psi_\pm$, such that each building block *on its own* would have an anomaly in the $U(1)$ symmetry which sends $\Psi_s \to e^{i\xi_s}\Psi_s,\ s=\pm$. The second property (already noted above) is that the two building blocks $\Psi_\pm$ couple only to the linear combinations $eA_\mu \pm B_\mu$ of vector fields. This property, combined with the axial anomaly of the Dirac fermion, is responsible for the form of the CSM response shown in Eq. (4.7.7). We now describe the construction of a bosonic theory with very similar properties.

Our low-energy theory for a BCSM in d dimensions (d even) consists of two copies of the $O(d+2)$ NLSM with WZ term, i.e., two copies of the boundary theory of the BIQH state in $d+1$ dimensions. To understand this system we briefly recall a few facts from Sec. 4.4 about the boundary theory of the BIQH state. The boundary of the BIQH state in $2m-1$ dimensions is described by an $O(2m)$ NLSM with WZ term. Here the dimension d is related to m by $d = 2m-2$ as we are constructing a model using the boundary theory for the BIQH state. Finally, recall that under a $U(1)$ transformation the NLSM field transforms as in Eq. (4.3.13) (in units where the boson charge e is set to 1). We showed that the properly gauged boundary action had an anomaly in this $U(1)$ symmetry, with the anomaly given explicitly by Eq. (4.4.33).

To construct an effective theory for a bosonic semi-metal in d dimensions we use two copies of the boundary theory of the BIQH state. We label the fields of the two copies by $\mathbf{n}_\pm$, or $b_{\ell,\pm}$ when written in terms of bosons, and we take the two copies to have opposite level on their WZ term, $k_\pm = \pm k$. Finally, in the effective theory we model the separation of the two copies in momentum/energy space by coupling the fields $b_{\ell,\pm}$ to the linear combinations $A_\mu \pm B_\mu$ of the external $U(1)$ gauge field A_μ and the momentum/energy shift field B_μ. Then our action for the BCSM theory takes the form

$$\tilde{S}_{BCSM}[\mathbf{n}_+,\mathbf{n}_-,A,B] = S_{gauged}[\mathbf{n}_+, A+B] + S_{gauged}[\mathbf{n}_-, A-B] \ , \tag{4.7.11}$$

where $S_{gauged}[\mathbf{n}, A]$ is the properly gauged action for one $O(d+2)$ NLSM with WZ term and coupled to the external field A (as constructed in Sec. 4.4). We put a tilde on $\tilde{S}_{BCSM}[\mathbf{n}_+,\mathbf{n}_-,A,B]$ because, as we now discuss, this action has an inconsistency and must be modified.

Suppose that the vector field B_μ, which is a constant in the context of the chiral semi-metal, instead had a non-trivial spacetime dependence, i.e., $dB \neq 0$. In this case the action in Eq. (4.7.11) is *not* invariant under the $U(1)$ gauge transformation $b_{\ell,\pm} \to e^{i\theta} b_{\ell,\pm}$, $A \to A + d\theta$. Instead, under this transformation one can show that the change in the action of Eq. (4.7.11) is

$$\delta_\theta \tilde{S}_{BCSM}[\mathbf{n}_+, \mathbf{n}_-, A, B] = -\frac{k}{(2\pi)^{m-1}} \sum_{p=0}^{m-1} \binom{m-1}{p} [1 + (-1)^{m-p}] \int_{\mathbb{R}^{d-1,1}} d\theta \wedge (dA)^p \wedge B \wedge (dB)^{m-2-p} \,. \tag{4.7.12}$$

where $2m - 1 = d + 1$. This equation requires some explanation. To compute it we used the relation Eq. (4.4.32) for the $U(1)$ anomaly for each gauged WZ theory in Eq. (4.7.11) (but coupled to the combinations of fields $A \pm B$ instead of A alone), then expanded the powers $(dA \pm dB)^{m-1}$ using the binomial expansion, and finally performed an integration by parts to move one derivative off of B and onto θ.

So in the presence of a spacetime-dependent B_μ, our putative semi-metal model is not invariant under $U(1)$ gauge transformations. To remedy this we modify the action by adding the counterterm

$$S_{ct}[A, B] = \frac{k}{(2\pi)^{m-1}} \sum_{p=0}^{m-1} \binom{m-1}{p} [1 + (-1)^{m-p}] \int_{\mathbb{R}^{d-1,1}} A \wedge (dA)^p \wedge B \wedge (dB)^{m-2-p} \,. \tag{4.7.13}$$

The change in this counterterm under $A \to A + d\theta$ exactly compensates for the change in Eq. (4.7.11) under the $U(1)$ gauge transformation, and so the modified BCSM action

$$S_{BCSM}[\mathbf{n}_+, \mathbf{n}_-, A, B] = \tilde{S}_{BCSM}[\mathbf{n}_+, \mathbf{n}_-, A, B] + S_{ct}[A, B] \,, \tag{4.7.14}$$

is completely gauge-invariant even in the presence of a spacetime-dependent B_μ. The counterterm $S_{ct}[A, B]$ is the analogue in our bosonic theory of the *Bardeen counterterm* that one adds to the theory of a Dirac fermion coupled to vector and axial vector gauge fields to ensure conservation of the vector current in the quantum theory[198]. Since this counterterm is absolutely necessary for the more general case of a spacetime-dependent B_μ, we argue that one should include it even in the simple semi-metal setting in which we take B_μ to be a constant. If we now restrict to the case of a constant B_μ, then only the $p = m - 2$ term in the counterterm survives, and the counterterm reduces to

$$S_{ct}[A, B] \to -\frac{2k}{(2\pi)^{m-1}} (m-1) \int_{\mathbb{R}^{d-1,1}} B \wedge A \wedge (dA)^{m-2} \,, \tag{4.7.15}$$

where we used $\binom{m-1}{m-2} = m - 1$.

To compute the response of the modified BCSM theory in Eq. (4.7.14), we attempt to remove the coupling to B

from the action via the chiral transformation

$$b_{\ell,\pm} \to e^{\pm i\xi} b_{\ell,\pm} \ , \tag{4.7.16}$$

where $\xi = B_\mu x^\mu$ as in the fermionic case. Note that this transformation takes the opposite sign for the two copies of the NLSM theory: this is the analogue in the bosonic theory of the chiral transformation of Eq. (4.7.4) that we performed in the fermionic case. Using the $U(1)$ anomaly for the boundary theory of the BIQH state from Eq. (4.4.33), we find that under this transformation the original effective action for the BCSM state transforms as

$$\tilde{S}_{BCSM}[\mathbf{n}_+, \mathbf{n}_-, A, B] \to \tilde{S}_{BCSM}[\mathbf{n}_+, \mathbf{n}_-, A, 0] + \tilde{S}^{(b)}_{eff}[A, B] \ , \tag{4.7.17}$$

where

$$\tilde{S}^{(b)}_{eff}[A, B] = -\frac{2k}{(2\pi)^{m-1}} \int_{\mathbb{R}^{d-1,1}} B \wedge A \wedge (dA)^{m-2} \ . \tag{4.7.18}$$

However, this is not the end of the story as the full action for the BCSM state contains the counterterm $S_{ct}[A, B]$. When we combine Eq. (4.7.18) with the counterterm (neglecting those parts of the counterterm containing dB), then we obtain the final expression for the response of the BCSM,

$$S^{(b)}_{eff}[A, B] = -2km \left(\frac{e}{2\pi}\right)^{m-1} \int_{\mathbb{R}^{d-1,1}} B \wedge A \wedge (dA)^{m-2} \ , \tag{4.7.19}$$

or in terms of d,

$$S^{(b)}_{eff}[A, B] = -2k \left(\frac{d}{2} + 1\right) \left(\frac{e}{2\pi}\right)^{\frac{d}{2}} \int_{\mathbb{R}^{d-1,1}} B \wedge A \wedge (dA)^{\frac{d}{2}-1} \ , \tag{4.7.20}$$

where we have restored the charge e of the bosons. This equation is the final form of the response of our BCSM model.

If we set $k = 1$ and compare the BCSM response in Eq. (4.7.20) to the fermionic CSM response in Eq. (4.7.7), then we see that the response of the BCSM in d dimensions is larger by a factor of $\left(\frac{d}{2} + 1\right)!$. To understand this number recall that our BCSM model in d dimensions is constructed from two copies of the boundary state for a BIQH state in $d + 1$ dimensions. Setting $d + 1 = 2m - 1$, we see that $\left(\frac{d}{2} + 1\right)! = m!$, so we find that the coefficients for the response of the bosonic and fermionic semi-metals in d dimensions differ by exactly the same factor we found in Sec. 4.4 for the coefficients for the response of BIQH and FIQH states in one dimension higher.

We can also see from Eq. (4.7.20) that *at the level of the electromagnetic response*, the BCSM theory at level k is equivalent to k copies of the BCSM theory at level 1. However, as a quantum field theory we certainly expect the theory at level k to be distinct from k copies of the theory at level 1. This can be seen very clearly in the case where $d = 2$. In this case the BCSM model consists of two copies of an $O(4)$ NLSM with WZ terms at levels k and

$-k$, respectively. The $O(4)$ NLSM can be reformulated in terms of a 2×2 $SU(2)$ matrix field, and so (when the coupling constant for the NLSM takes on a particular value), the $O(4)$ NLSM with WZ term at level k is equivalent to the $SU(2)_k$ Wess-Zumino-Witten conformal field theory. Now the $SU(2)_k$ theory is distinct from k copies of the $SU(2)_1$ theory (this can be seen by comparing central charges), and so we conclude that even in the simplest case of two dimensions, the BCSM model at level k is distinct (as a quantum field theory) from k copies of the BCSM model at level 1. However, it is entirely possible that k copies of the BCSM model at level 1 could flow under the Renormalization Group to the BCSM model at level k. In the simple $d = 2$ case discussed in this paragraph this flow is allowed by Zamolodchikov's c-theorem[199].

4.7.2 Bosonic analogue of the parity anomaly on the boundary of BTI phases

In this subsection we show that the half-quantized BIQH on the BTI boundary, which we derived in Sec. 4.6, represents a bosonic analogue of the parity anomaly[58–60, 70, 165], which is an anomaly that is usually associated to massless Dirac fermions in odd dimensions. To start, we give a brief review of the parity anomaly in the fermionic case before explaining the bosonic analogue.

To understand the parity anomaly for Dirac fermions in odd dimensions, consider a theory of a single massless Dirac fermion Ψ with $U(1)$ symmetry in $2m-1$ dimensions. We can couple Ψ to an external electromagnetic field A and then integrate out Ψ to obtain the partition function

$$Z[A] = \int [D\Psi][D\overline{\Psi}] e^{iS[\Psi,A]} \ , \tag{4.7.21}$$

and the effective action for the external field A,

$$S_{eff}[A] = -i \ln(Z[A]) \ . \tag{4.7.22}$$

The action $S[\Psi, A]$ (with Ψ a massless fermion) has an additional discrete symmetry, which is time-reversal symmetry when the spacetime dimension equals 3 mod 4, or charge-conjugation (particle-hole) symmetry[76] when the spacetime dimension equals 1 mod 4 (in Euclidean spacetime the discrete symmetry in any dimension can be chosen to be reflection of a single coordinate). However, when one proceeds to calculate the effective action $S_{eff}[A]$, one finds that there is no choice of regularization procedure which yields an action $S_{eff}[A]$ which has this extra discrete symmetry and is also gauge-invariant. In other words, one can choose to preserve either the discrete symmetry, or gauge invariance, but not both. For example, when Pauli-Villars regularization is used to compute $S_{eff}[A]$, the mass terms for the regulator fermions explicitly break the discrete symmetry, and this results in the appearance of a term in $S_{eff}[A]$ which also breaks the discrete symmetry.

At this point it helps to look at a specific example. We choose the case of a massless Dirac fermion Ψ in three spacetime dimensions with $U(1)$ symmetry and $\mathbb{Z}_2^T$ time-reversal symmetry, which was the case originally studied in Refs. [58, 59]. This case also applies directly to the study of the surface of the familiar electron topological insulator in four dimensions. Because of the $U(1)$ symmetry, Ψ can be coupled to the external field A. To discuss the transformation of Ψ under time-reversal, it is convenient[141] to choose the gamma matrices in the "mostly minus" metric to be $\gamma^0 = \sigma^z$, $\gamma^1 = i\sigma^y$ and $\gamma^2 = -i\sigma^x$, where σ^a, $a = x, y, z$, are the three Pauli matrices (and recall that a single Dirac fermion in three dimensions has two components). With this choice, the time-reversal transformation of Ψ takes the form

$$\mathbb{Z}_2^T : \ \Psi(t, \mathbf{x}) \to i\sigma^y \Psi(-t, \mathbf{x}) \ , \tag{4.7.23}$$

while the components A_μ of A transform as

$$\mathbb{Z}_2^T : \ A_0(t, \mathbf{x}) \quad \to \quad A_0(-t, \mathbf{x}) \tag{4.7.24}$$

$$A_i(t, \mathbf{x}) \quad \to \quad -A_i(-t, \mathbf{x}) \ , \ i = 1, 2 \ . \tag{4.7.25}$$

In the absence of a mass term for Ψ the action $S[\Psi, A]$ for Ψ minimally coupled to A has time-reversal symmetry in addition to the $U(1)$ symmetry. However, when Pauli-Villars regularization is used to compute $S_{eff}[A]$, one finds that $S_{eff}[A]$ contains the time-reversal-breaking CS term for A[5]. In addition, the level of this CS term is equal to $\pm\frac{1}{2}$, which is half of the minimum Hall conductance possible for free fermions in three dimensions (i.e., the CS term with level $\pm\frac{1}{2}$ is like a half-quantized Integer Quantum Hall state of fermions). One can think of the parity anomaly as a quantum version of the spontaneous breaking of a discrete symmetry. Indeed, the value of the induced CS term in $S_{eff}[A]$ is determined by the sign of the mass of the Pauli-Villars regulator fermion, and this choice of sign is arbitrary in the same way that the choice of a particular point on the vacuum manifold of a "Mexican hat" potential is arbitrary.

To demonstrate that a bosonic analogue of the parity anomaly occurs in the boundary of a BTI phase, we first need to discuss the symmetries of the BTI theory coupled to A. As we discussed in Sec. 4.3, the NLSM field n_a transforms under the $\mathbb{Z}_2^T$ or $\mathbb{Z}_2^C$ symmetry of the BTI as shown in Eq. (4.3.14). Recall that in the case where the $\mathbb{Z}_2$ symmetry is time-reversal, we also need to send $t \to -t$ in the argument of the NLSM field n_a and in the action. For a spacetime of dimension d (which in our convention is the dimension of the boundary of the SPT phase) the field A transforms

[5]In a more precise treatment Pauli-Villars regularization leads to an effective action which is proportional to the Atiyah-Potodi-Singer *eta invariant* of the Dirac operator[60]. In certain cases the expression in terms of the eta invariant can then be replaced with the simpler expression in terms of a CS term with half-quantized level. However, this more precise treatment using the eta invariant still gives an effective action that breaks the time-reversal symmetry of the original Lagrangian for Ψ and A.

under time-reversal and charge-conjugation as

$$\mathbb{Z}_2^T : \ A_0(t,\mathbf{x}) \to A_0(-t,\mathbf{x}) \tag{4.7.26}$$

$$A_i(t,\mathbf{x}) \to -A_i(-t,\mathbf{x}) \ , \ i = 1,\dots,d-1 \ , \tag{4.7.27}$$

and

$$\mathbb{Z}_2^C : \ A_\mu(t,\mathbf{x}) \to -A_\mu(t,\mathbf{x}) \ , \ \forall\, \mu \ , \tag{4.7.28}$$

where $\mathbf{x} = (x^1,\dots,x^{d-1})$ denotes the spatial coordinates.

The gauged WZ action of Eq. (4.6.24) for the boundary of the BTI phase has the $\mathbb{Z}_2^C$ (for m odd) or $\mathbb{Z}_2^T$ (for m even) symmetry of the BTI, in addition to the invariance under $U(1)$ gauge transformations. To see this we simply note that the counterterms from Eq. (4.6.25) transform under these two $\mathbb{Z}_2$ symmetries as

$$\mathbb{Z}_2^T : \ S_{ct}^{(r)}[\mathbf{n},A] \to (-1)^m S_{ct}^{(r)}[\mathbf{n},A] \ , \tag{4.7.29}$$

and

$$\mathbb{Z}_2^C : \ S_{ct}^{(r)}[\mathbf{n},A] \to (-1)^{m+1} S_{ct}^{(r)}[\mathbf{n},A] \ . \tag{4.7.30}$$

So the gauged WZ action for the BTI boundary has $\mathbb{Z}_2^T$ symmetry for m even and $\mathbb{Z}_2^C$ symmetry for m odd.

Now, although the gauged WZ action for the BTI boundary has the $\mathbb{Z}_2$ symmetry, we have seen in Sec. 4.6 that it is possible to add the symmetry-allowed term $\mu\, n_{2m+1}^2$ to the Lagrangian and drive the boundary of the BTI into a phase in which the $\mathbb{Z}_2$ symmetry is *spontaneously* broken by the condensate $n_a = \pm\delta_{a,2m+1}$. In addition, we showed that when the field n_a condenses in this way it leads to a CS term in the effective action for A on the boundary of the BTI phase. The CS term in $2m-1$ dimensions breaks $\mathbb{Z}_2^T$ symmetry for m even, and $\mathbb{Z}_2^C$ symmetry for m odd, so the effective action for A does not have the $\mathbb{Z}_2$ symmetry of the BTI phase. We also saw that the CS level turned out to be quantized in *half-integer* multiples of $m!$.

Since the CS term in the effective action for the boundary breaks the $\mathbb{Z}_2$ symmetry of the BTI phase, and since the boundary also exhibits a "half" BIQH response, we conclude that the boundary theory of the BTI phase exhibits an anomaly in the $\mathbb{Z}_2$ symmetry which is almost an exact analogue of the parity anomaly of a Dirac fermion in odd dimensions.

There is, however, one important difference between the bosonic analogue of the parity anomaly discussed here and the original parity anomaly for Dirac fermions. The difference is the fact that in the bosonic case the spontaneous breaking of the $\mathbb{Z}_2$ symmetry is a classical effect, whereas in the original fermionic case the $\mathbb{Z}_2$ symmetry is broken spontaneously only at the quantum level (by the choice of the sign of the mass of the regulator fermions). One likely explanation for this difference is as follows. Since the NLSM description of the bosonic SPT phase is an effective field theory description, i.e., it does not involve the microscopic degrees of freedom in the SPT phase, it is entirely possible that the quantum anomaly of any putative microscopic description of the SPT phase is already captured at the classical level in the effective NLSM description of the phase. This is, in fact, exactly the way in which quantum anomalies of fermionic theories are captured at the classical level in effective descriptions of those fermionic theories in terms of gauged WZ actions[37, 38]. In addition, we have already seen in this Chapter that the pertubative $U(1)$ anomaly on the boundary of BIQH states is completely captured at the classical level in the gauged WZ description of the BIQH boundary. For this reason we believe that the bosonic analogue of the parity anomaly discussed here is a bona-fide quantum effect that occurs in the boundary theory of a BTI phase, and that this anomaly would appear as a true quantum anomaly in a more microscopic description of the boundary of the BTI. We are therefore led to argue that, due to this anomaly, the boundary theory of a $2m$-dimensional BTI phase cannot be realized intrinsically in $2m-1$ dimensions without breaking either the $U(1)$ or the $\mathbb{Z}_2$ symmetry of the BTI phase.

Finally, let us describe one more way of understanding the bosonic analogue of the parity anomaly in the specific case of the boundary theory of the four-dimensional BTI. As we know, the boundary theory is an $O(5)$ NLSM with WZ term in three dimensions. Let us investigate what happens in this theory when we thread a 2π delta function flux of the gauge field at a point in space. This method of analysis in known to expose the parity anomaly in the theory of a single massless Dirac fermion in three dimensions (see, for example, Ref. [147]) and so it should work in this case as well. For simplicity we consider a deformation of the $O(5)$ theory in which we set $n_5 = 0$ (this deformation preserves the $U(1)$ and time-reversal symmetry). In this case the WZ term at level k reduces to a theta term for the four component field $(n_1, \ldots, n_4)$ with the theta angle equal to $\theta = \pi k$. In particular, for $k = 1$ (which represents the non-trivial BTI phase in four dimensions) the WZ term with level $k = 1$ reduces to a theta term with theta angle $\theta = \pi$.

Threading a 2π delta function flux at a point $\mathbf{x}_0$ in space will cause the phase of both bosons $b_1 = n_1 + in_2$ and $b_2 = n_3 + in_4$ to wind by 2π about $\mathbf{x}_0$, i.e., there is a vortex centered at $\mathbf{x}_0$ in the phase of both bosons. In Appendix B of Ref. [34], two of us performed a detailed study of vortex configurations of a *single* boson b_1 or b_2 in the $O(4)$ NLSM with theta term and with $\theta = \pi$. In particular we quantized global fluctuations of the theory over such a vortex background and showed that the ground state of these fluctuations was doubly degenerate, with the two degenerate states having charges $\pm\frac{1}{2}$. This analysis confirmed the arguments of Ref. [19] that a vortex in a single boson b_1 or b_2 should carry charge $\pm\frac{1}{2}$. As stated above, threading a 2π flux of A_μ at $\mathbf{x}_0$ induces a vortex in *both* b_1 and b_2 at that point. This composite object has an integer charge and so is naively gauge-invariant, however, this composite object can actually be shown to be a fermion[19, 34, 200]. This fact clearly shows the anomaly in theory, as there should be no local fermionic particle with integer charge in a system made up of bosons of unit charge. The existence of a fermion with unit charge in the boundary theory of the BTI can also be inferred from the presence of a CS term at level $N_3 = 1$ in the response action for the BTI boundary using an argument from Ref. [18].

4.7.3 $\mathbb{Z}_2$ symmetry-breaking domain walls on the boundary of BTI

We close this section by analyzing the physics of a domain wall between two opposite $\mathbb{Z}_2$ symmetry-breaking regions on the boundary of a BTI state in $2m$ dimensions. In particular, we derive the low-energy theory that exists on the domain wall, and we show that this theory has a $U(1)$ anomaly which is exactly canceled by the contributions of the CS response actions for the $\mathbb{Z}_2$ symmetry-breaking regions on either side of the domain wall. We find that the theory which lives on the domain wall is identical to the boundary theory for the BIQH phase in $2m-1$ dimensions, and so this demonstration of anomaly cancellation for domain wall configurations on the BTI boundary provides a nice consistency check between our gauged actions for BIQH and BTI surfaces.

To start, recall from Sec. 4.6 that the boundary of a BTI phase in $2m$ dimensions, which is described by an $O(2m+1)$ NLSM with WZ term at level k, can exhibit a $\mathbb{Z}_2$ symmetry-breaking response of the form ($d = 2m-1$ is again the boundary dimension)

$$S_{eff}[A] = \pm\frac{1}{2}\frac{k}{(2\pi)^{m-1}}\int_{\mathbb{R}^{d-1,1}} A\wedge F^{m-1} , \tag{4.7.31}$$

when the NLSM field n_a condenses as in Eq. (4.6.28), i.e., $n_{2m+1} = \pm 1$ and all other components of $\mathbf{n}$ equal to zero. As we discussed earlier, this particular condensation pattern is the only one which preserves the $U(1)$ symmetry of the BTI phase while breaking the $\mathbb{Z}_2$ symmetry.

We now consider a domain wall configuration between opposite $\mathbb{Z}_2$ breaking regions on the boundary. Let $(x^0,\dots,x^{d-1})$ be the boundary spacetime coordinates. We study a configuration of the system in which n_{2m+1} condenses as $n_{2m+1} = 1$ in the region $x^{d-1} > 0$, and as $n_{2m+1} = -1$ in the region $x^{d-1} < 0$. Hence, the domain wall is in the x^{d-1} direction. Then on the two sides of the domain wall the electromagnetic response is given by

$$S_{eff,+}[A] = \frac{1}{2}\frac{k}{(2\pi)^{m-1}}\int_{\mathbb{H}_+^{d-1,1}} A\wedge F^{m-1} , \tag{4.7.32}$$

and

$$S_{eff,-}[A] = -\frac{1}{2}\frac{k}{(2\pi)^{m-1}}\int_{\mathbb{H}_-^{d-1,1}} A\wedge F^{m-1} , \tag{4.7.33}$$

respectively, where $\mathbb{H}_+^{d-1,1}$ denotes the half space $\{x\in\mathbb{R}^{d-1,1} |\, x^{d-1} > 0\}$, and similarly for $\mathbb{H}_-^{d-1,1}$. If we perform a gauge transformation then the change in the total effective action is

$$\delta_\xi S_{eff,+}[A] + \delta_\xi S_{eff,-}[A] = \frac{k}{(2\pi)^{m-1}} \int_{\mathbb{R}^{d-2,1}} \xi F^{m-1} \;, \tag{4.7.34}$$

where the integration is over the domain wall which is just the space $\mathbb{R}^{d-2,1}$ sitting at $x^{d-1} = 0$. Note also that the contributions from $S_{eff,\pm}[A]$ add instead of subtract due to the fact that the domain wall is on the opposite side of the two regions (the domain wall lies to the right of the region $\mathbb{H}^{d-1,1}_+$ and to the left of the region $\mathbb{H}^{d-1,1}_-$, so when we integrate the total derivative the boundary terms coming from each integral appear with opposite signs).

Next we derive the theory which lives on the domain wall and show that it has a $U(1)$ anomaly which precisely cancels the gauge transformation from Eq. (4.7.34). To derive this theory we analyze the BTI surface theory in the presence of a domain wall in n_{2m+1}. Concretely, we assume that the $O(2m+1)$ NLSM field takes on the domain wall configuration,

$$\mathbf{n} = \{\sin(f(x^{d-1}))\mathbf{N}(x^0, \dots, x^{d-2}), \cos(f(x^{d-1}))\} \;, \tag{4.7.35}$$

where $\mathbf{N}$ is a $2m$-component unit vector which depends only on the coordinates $(x^0, \dots, x^{d-2})$ on the domain wall, and where $f(x^{d-1})$ is a function with the limiting behavior

$$\lim_{x^{d-1}\to\infty} f(x^{d-1}) \quad = \quad 0 \tag{4.7.36}$$

$$\lim_{x^{d-1}\to-\infty} f(x^{d-1}) \quad = \quad \pi \;. \tag{4.7.37}$$

This guarantees that $\mathbf{n}$ asymptotes to a configuration with $n_{2m+1} = \pm 1$ as $x^{d-1} \to \pm\infty$. To solve for the theory which lives on the domain wall, we evaluate the $O(2m+1)$ NLSM action (with WZ term) on this configuration. Evaluating the kinetic term of the NLSM on the domain wall configuration is simple, provided that we assume the function $f(x^{d-1})$ is sufficiently well-behaved so that the integration over x^{d-1} gives a finite answer. We therefore focus our attention on the WZ term since any anomalous behavior of the domain wall theory should come from this term. The WZ term involves an extension $\tilde{\mathbf{n}}$ of the field $\mathbf{n}$ into a fictitious extra direction with coordinate $s \in [0,1]$, and so we need to decide how to extend our domain wall configuration into this extra direction. Here we assume the extension takes the form

$$\tilde{\mathbf{n}} = \{\sin(f(x^{d-1}))\tilde{\mathbf{N}}(s, x^0, \dots, x^{d-2}), \cos(f(x^{d-1}))\} \;, \tag{4.7.38}$$

so that all of the s-dependence of the extension is in the extended $2m$-component field $\tilde{\mathbf{N}}$, while the function $f(x^{d-1})$ still depends only on x^{d-1}.

We now examine how the WZ term of the $O(2m+1)$ NLSM reduces on the extended domain wall configuration $\tilde{\mathbf{n}}$ of Eq. (4.7.38). The analysis is similar (but not identical) to that in Appendix C.4 for the dimensional reduction of theta terms in NLSMs on defect configurations of the NLSM field. Recall that the WZ term takes the form

$$S_{WZ}[\mathbf{n}] = \frac{2\pi k}{\mathcal{A}_{2m}} \int_{[0,1]\times\mathbb{R}^{d-1,1}} \tilde{\mathbf{n}}^* \omega_{2m}, \tag{4.7.39}$$

where ω_{2m} is the volume form for the sphere S^{2m}, and $[0,1]\times\mathbb{R}^{d-1,1}$ is the extended spacetime (which we called $\mathcal{B}$ before). Using the relations

$$dn_a = \sin(f)dN_a + \cos(f)N_a df\ ,\ a = 1,\dots,2m\ , \tag{4.7.40}$$

and

$$dn_{2m+1} = -\sin(f)df\ , \tag{4.7.41}$$

one can show that on this configuration the volume form ω_{2m} for $\mathbf{n}$ reduces to

$$\omega_{2m} \to [\sin(f)]^{2m-1} df \wedge \omega_{2m-1}\ , \tag{4.7.42}$$

where

$$\omega_{2m-1} = \sum_{a=1}^{2m} (-1)^{a-1} N_a dN_1 \wedge \cdots \wedge \overline{dN_a} \wedge \cdots \wedge dN_{2m}\ , \tag{4.7.43}$$

is the volume form for N_a. Since the WZ term involves the pullback $\tilde{\mathbf{n}}^*\omega_{2m}$ of the volume form to the extended spacetime, we find that the WZ term reduces as

$$\begin{aligned} S_{WZ}[\mathbf{n}] &\to \frac{2\pi k}{\mathcal{A}_{2m}} \int_{-\infty}^{\infty} dx^{d-1}\ f'(x^{d-1})[\sin(f(x^{d-1}))]^{2m-1} \int_{[0,1]\times\mathbb{R}^{d-2,1}} \tilde{\mathbf{N}}^*\omega_{2m-1} \\ &= \frac{2\pi k}{\mathcal{A}_{2m}} \left(-\frac{\sqrt{\pi}\,\Gamma(m)}{\Gamma(m+\frac{1}{2})} \right) \int_{[0,1]\times\mathbb{R}^{d-2,1}} \tilde{\mathbf{N}}^*\omega_{2m-1} \\ &= -\frac{2\pi k}{\mathcal{A}_{2m-1}} \int_{[0,1]\times\mathbb{R}^{d-2,1}} \tilde{\mathbf{N}}^*\omega_{2m-1}\ . \end{aligned} \tag{4.7.44}$$

We see that the theory localized on the domain wall is an $O(2m)$ NLSM for the field $\mathbf{N}$, with a WZ term at level $-k$. This theory also appears at the boundary theory of the BIQH phase in $2m-1$ dimensions, as discussed in Sec. 4.4. The extra minus sign on the level of the domain wall theory, as compared with the level of the boundary theory of the BTI, is very important. Indeed, from our previous formula Eq. (4.4.32) for the $U(1)$ anomaly of the $O(2m)$ NLSM with WZ term we see that, under a gauge transformation, the theory on the domain wall transforms as

$$\delta_\xi S_{DW}[\mathbf{N}, A] = -\frac{k}{(2\pi)^{m-1}} \int_{\mathbb{R}^{d-2,1}} \xi F^{m-1}\ . \tag{4.7.45}$$

This exactly cancels Eq. (4.7.34), which was the contribution flowing into the domain wall from the $\mathbb{Z}_2$ breaking regions on either side, and so this calculation gives a nice example of anomaly inflow at the domain walls on the boundary of SPT phases. It also provides an important consistency check of the gauged WZ actions calculated in this Chapter for the boundaries of BIQH and BTI phases (since it relates the calculation of the BTI boundary CS response to the calculation of the anomaly of the BIQH boundary theory).

4.8 Conclusion

In this Chapter we calculated the electromagnetic response of bosonic SPT phases with $U(1)$ symmetry in all spacetime dimensions. In particular, we focused our attention on BIQH phases in odd dimensions and BTI phases in even dimensions. To calculate the response of these phases we used the NLSM description of bosonic SPT phases and the tool of gauged WZ actions. This enabled us to compute the coefficients N_{2m-1} and Θ_{2m} which determine, via Eqs. (4.2.1) and (4.2.2), the electromagnetic response of BIQH and BTI states in all odd and even spacetime dimensions, respectively. We found that for BIQH states the coefficient N_{2m-1} was quantized in units of $m!$, and that the non-trivial BTI state in $2m$ dimensions has $\Theta_{2m} = 2\pi\left(\frac{m!}{2}\right)$. This response for the BTI is equivalent to a $\mathbb{Z}_2$ symmetry-breaking Quantum Hall state on the boundary of the BTI with CS level equal to $\frac{m!}{2}$, which is exactly half the response of the BIQH state which can be realized intrinsically in the same spacetime dimension. In Sec. 4.5 we showed that the value of $m!$ for the CS level can be understood by studying the transformation of the CS term under large $U(1)$ gauge transformations on general Euclidean manifolds which may or may not admit a spin structure. In that section we also applied this gauge invariance argument to study the electromagnetic *and* gravitational responses of fermionic SPT phases with $U(1)$ symmetry in odd spacetime dimensions.

We then used our gauged WZ actions for the boundaries of the BIQH and BTI phases to further investigate the physics of BIQH and BTI states, and to construct other interesting states. In particular, we showed how two copies of the BIQH boundary theory could be used to construct an effective theory for a bosonic analogue of a Weyl, or chiral, semi-metal (a "bosonic chiral semi-metal" or BCSM state) in any even spacetime dimension. We also showed that the boundary of the BTI state exhibits a bosonic analogue of the parity anomaly of a Dirac fermion in odd dimensions, and we used this fact to argue that the boundary theory of the BTI in $2m$ dimensions cannot be realized intrinsically in $2m-1$ dimensions while preserving the symmetry of the BTI state. We also explored the phenomenon of anomaly inflow at $\mathbb{Z}_2$ symmetry-breaking domain walls on the boundaries of BTI states.

From a technical point of view one of the most interesting results of the Chapter is our explicit construction of gauged WZ actions for $O(2m)$ and $O(2m+1)$ NLSMs, and with the gauge group $U(1)$. This construction allowed us to overcome the difficulties associated with calculating the electromagnetic response of bosonic SPT phases from their

NLSM description. In addition, as we reviewed in Appendix C.1, the problem of constructing a gauged WZ action is equivalent to the mathematical problem of constructing equivariant extensions of the volume form on the target space of the NLSM. Then from a mathematical point of view we can say that we have succeeding in constructing an equivariant extension of the volume form ω_{2m} on S^{2m} (this is the BTI case), whereas in the case of S^{2m-1} we have constructed an extension of ω_{2m-1} which is almost, but not quite, equivariantly closed (this is the BIQH case). The fact that we could not construct an equivariant extension of ω_{2m-1} is mathematically equivalent to the statement that the boundary theory of the BIQH phase has a perturbative anomaly in the $U(1)$ symmetry, as we expect based on physical arguments.

Our work in this Chapter opens up many possible directions for future investigations. In particular, it would be nice to have a physical argument along the lines of the one in Ref. [18] for why the CS level for the BIQH phase is quantized in units of $m!$ in $2m-1$ dimensions. Perhaps one can find a physical argument for this quantization by studying generalized braiding processes of extended excitations in gapped bosonic phases in higher dimensions, but we do not have any concrete suggestions as to which excitations and braiding processes might be relevant. Another possible direction would be to apply the general method of gauging WZ actions from Ref. [41] to compute the "electromagnetic" response of SPT phases with the symmetry of a non-Abelian Lie group G. From a mathematical point of view it would also be interesting to investigate whether the theory of G-equivariant cohomology over an appropriate target manifold could be used to classify SPT phases with the symmetry of a Lie group G. Finally, it would be interesting to use the bosonic analogue of the parity anomaly discussed in this Chapter as a guide to investigate possible dual descriptions of the boundary of BTI phases in all dimensions, analogous to the recent investigations into the dual description of the boundary of the electron topological insulator and BTI in four spacetime dimensions[34, 141–148].

Chapter 5

Perturbative and global anomalies in bosonic analogues of integer quantum Hall and topological insulator phases[1]

5.1 Introduction

In the past few years it was realized that a powerful way to understand symmetry-protected topological (SPT) phases with symmetry group G in d (spacetime) dimensions is to study *'t Hooft anomalies* of $(d-1)$-dimensional theories with global G-symmetry [66–69]. A theory with global G-symmetry has a 't Hooft anomaly if it cannot be consistently coupled to a background gauge field A for the symmetry group G [21]. It is often the case that an anomalous $(d-1)$-dimensional theory can be realized in a gauge invariant manner at the boundary of a d-dimensional SPT phase. In that case, the anomaly of the boundary theory is canceled by the gauge variation of the bulk effective action for the SPT phase. This cancellation mechanism is known as *anomaly inflow* [75]. It is likely that all bulk-boundary correspondences in SPT phases can be understood through some version of the anomaly inflow mechanism, but perhaps involving global anomalies instead of the perturbative anomalies originally studied in Ref. [75].

It is clear from the discussion above that characterizing boundary anomalies offers a precise way to understand the bulk-boundary correspondence in SPT phases, topological insulators, and related systems. For example, the presence of a single chiral fermion at the edge of the $\nu = 1$ integer quantum Hall state in $2+1$ dimensions (and also the single chiral boson at the edge of the Laughlin states) can be understood very simply using anomaly inflow arguments [73, 74]. This chiral fermion is needed to cancel the gauge variation of the bulk Chern-Simons term[2]

$$S_{CS}[A] = \frac{1}{4\pi}\int_X A \wedge dA \ , \tag{5.1.1}$$

which describes the response of the integer quantum Hall state to an external electromagnetic field $A = A_\mu dx^\mu$. We also note that anomaly inflow has been discussed for analogs of the integer quantum Hall state in all odd spacetime dimensions [52].

A related, but much more subtle, example of anomaly inflow occurs in time-reversal invariant, free-fermion topo-

[1]This Chapter is reproduced from Matthew F. Lapa and Taylor L. Hughes, Phys. Rev. B 96, 115123 (2017). ©2017 American Physical Society. This paper is also cited as Ref. [61] in the References section of the thesis.

[2]We use differential form notation and work in a system of units where $\hbar = e = c = 1$.

logical insulators in $3{+}1$ dimensions [7, 8]. In Ref. [70] Witten has shown (among other results) that the bulk-boundary correspondence in this system can be understood very precisely in terms of the parity anomaly of a Dirac fermion with $U(1)$ and time-reversal symmetry in $2+1$ dimensions [58–60, 165]. The parity anomaly is intimately related to the Atiyah-Potodi-Singer index theorem [71, 201, 202] for the Dirac operator on an even-dimensional manifold with boundary (see Ref. [60] for the relation), and this connection was a central theme in Ref. [70]. The connection between the parity anomaly and the boundary theory of the topological insulator, and in particular the fact that the bulk and boundary together are gauge invariant, was also previously discussed in Ref. [203].

In a separate series of developments, bosonic analogues of the integer quantum Hall and topological insulator states were introduced and studied in detail in the SPT literature. The bosonic integer quantum Hall (BIQH) state is an SPT phase of bosons with $U(1)$ symmetry in $2+1$ dimensions [9, 18, 149–157]. It is characterized by a Hall conductance which is an *even* integer (in units of $\frac{e^2}{h}$). On the other hand, the bosonic topological insulator (BTI) state is an SPT phase of bosons with $U(1)$ symmetry and $\mathbb{Z}_2$ time-reversal symmetry in $3{+}1$ dimensions [19, 20, 158, 159]. It is characterized by a bulk electromagnetic response of the "Chern character" type

$$S_{CC}[A] = \frac{\Theta}{8\pi^2}\int_X F \wedge F \ , \tag{5.1.2}$$

with coefficient $\Theta = 2\pi$. In a recent work, the present authors computed the electromagnetic response of generalizations of the BIQH and BTI states to *all* odd and even spacetime dimensions, respectively [57].

Given these separate developments, a natural next step would be to give a precise characterization of the anomalies at the boundaries of the BIQH and BTI states. In Ref. [57] we initiated such a program. There we used a nonlinear sigma model (NLSM) description [35, 36, 160–163, 184–188] of the boundary of the BIQH state in odd dimensions to compute the perturbative $U(1)$ anomaly of the boundary theory. Our result implied that the electromagnetic response of the bulk of a BIQH state in $2m-1$ dimensions is characterized by a Chern-Simons term[3] with level $N_{2m-1} = (m!)k$, $k \in \mathbb{Z}$, where the value $k=1$ represents the fundamental BIQH state.

In Ref. [57] we also argued that the boundary theory of the $2m$-dimensional BTI state exhibits a bosonic analogue of the well-known parity[4] anomaly of Dirac fermions in three dimensions. Our argument was based on a demonstration (again using a NLSM description) that the boundary of the BTI state can exhibit a $\mathbb{Z}_2$ symmetry-breaking electromagnetic response described by a Chern-Simons term with level $N_{2m-1} = \frac{m!}{2}$ for the external field A. Since this boundary response is *half* the response of the fundamental BIQH state in $2m-1$ dimensions, we argued, by analogy with the case of a massless Dirac fermion (with Hall conductance = Chern-Simons level = $\frac{1}{2}$) on the surface

[3] See Eq. (5.2.1) for our normalization of the Chern-Simons term in $2m-1$ dimensions.

[4] As we explained in Ref. [57], in spacetime dimensions $2m$ with m *odd* the $\mathbb{Z}_2$ symmetry of the BTI state is a unitary charge-conjugation symmetry and not time-reversal symmetry. For these cases the word "parity" is not a very good description of the symmetry which is anomalous. However, for ease of presentation we will continue to refer to the global anomalies discussed here as "bosonic analogues of the parity anomaly".

of the $(3+1)$-dimensional topological insulator [7, 8], that the boundary theory of the BTI displays a bosonic analogue of the parity anomaly.

In this Chapter we continue this program of characterizing anomalies at the boundary of BIQH and BTI states. In the first part of the Chapter we revisit the perturbative $U(1)$ anomaly at the boundary of $(2m-1)$-dimensional BIQH states. In Ref. [57] we computed this anomaly by gauging the Wess-Zumino (WZ) term in an $O(2m)$ NLSM description of the boundary of the BIQH state. In any NLSM, the field is a map from spacetime to a manifold $\mathcal{M}$, known as the *target space* of the NLSM. In the $O(2m)$ NLSM the target space is just the $(2m-1)$-dimensional unit sphere S^{2m-1}, and the NLSM field $\mathbf{n}$ is a $2m$-component unit vector. This particular NLSM description possesses a $SO(2m)$ global symmetry, which is much larger than the $U(1)$ symmetry required to protect the BIQH state. One might then wonder if (perhaps) more realistic models of the BIQH boundary can be found which still possess the correct perturbative $U(1)$ anomaly, but have only the $U(1)$ global symmetry of the BIQH state. In this Chapter we show that a large family of such models do indeed exist by proving the following result.

> Let $\mathcal{M}$ be *any* $(2m-1)$-dimensional manifold which can be reached from S^{2m-1} by smooth deformations which preserve the $U(1)$ symmetry of the BIQH phase (i.e., we have a diffeomorphism $f : \mathcal{M} \to S^{2m-1}$ which is *equivariant* with respect to the $U(1)$ symmetry). Then a description of the boundary of the BIQH state using a NLSM with target space $\mathcal{M}$ has *the same* perturbative $U(1)$ anomaly as the $O(2m)$ NLSM description.

In the second part of the Chapter we revisit the bosonic analogue of the parity anomaly at the boundary of the BTI states. In the simplest case of the BTI state in two spacetime dimensions we are able to compute the partition function of the gauged boundary theory exactly. The BTI state in two dimensions has the symmetry group $G = U(1) \rtimes \mathbb{Z}_2$, where $\mathbb{Z}_2$ represents a unitary charge-conjugation symmetry. Our exact computation of the boundary partition function shows that the boundary of the BTI does indeed exhibit a bosonic analogue of the global anomaly of Dirac fermions in $0+1$ dimensions which also have $U(1)$ symmetry and $\mathbb{Z}_2$ charge-conjugation symmetry [76]. We first compute this anomaly within the $O(3)$ NLSM description (with target space S^2) of the BTI boundary which we previously used in Ref. [57]. Based on this calculation, one might again wonder if a more realistic model of the BTI boundary can be found which has the same global anomaly, but which possesses only the $G = U(1) \rtimes \mathbb{Z}_2$ symmetry of the BTI state and not the full $SO(3)$ symmetry of the $O(3)$ NLSM. We again show that such models do exist by proving the following result.

> Let $\mathcal{M}$ be *any* two-dimensional manifold which can be reached from S^2 by smooth deformations which preserve the full $G = U(1) \rtimes \mathbb{Z}_2$ symmetry of the BTI state (i.e., we have a diffeomorphism $f : \mathcal{M} \to S^2$ which

is *equivariant* with respect to the action of the group G). Then a description of the boundary of the BTI state using a NLSM with target space $\mathcal{M}$ has *the same* global anomaly as the $O(3)$ NLSM description.

To prove this result we use the powerful *equivariant localization* technique originally developed for the exact computation of certain phase space path integrals [204–208]. Whereas for the perturbative anomaly we are able to extend our proof to any spacetime dimension, the calculation for global anomalies becomes challenging in higher dimensions and is not easily extendable. We comment on this difficulty later, and discuss possible alternative approaches.

As in our previous work [57], gauged WZ actions play a central role in the calculations in this Chapter. Gauging WZ actions, and also obstructions to gauging these actions (i.e., anomalies), have been discussed previously in Refs. [37–41, 209–211]. Since we consider two kinds of anomalies in this Chapter (perturbative and global), it is important for us to explain at the outset how exactly our anomalies are related to obstructions to gauging a WZ action. For the perturbative $U(1)$ anomalies that we study, the anomaly that we find is a direct result of the existence of an obstruction to gauging the WZ action. Therefore, these anomalies are already present at the level of the classical action for these theories. On the other hand, for the global anomalies that we study there is *no obstruction* to gauging the $U(1)$ symmetry of the WZ action. Instead, the anomaly is a completely quantum effect which stems from an inability to regulate the theory in such a way as to preserve both large $U(1)$ gauge invariance, and the additional discrete $\mathbb{Z}_2$ symmetry of the theory.

This Chapter is organized as follows. In Sec. 5.2 we analyze perturbative $U(1)$ anomalies at the even-dimensional boundary of BIQH states in generic odd spacetime dimensions. In Sec. 5.3 we analyze the global anomaly at the $(0+1)$-dimensional boundary of the $(1+1)$-dimensional BTI state. In Sec. 5.4 we comment on the expected behavior of the boundary theories studied in this Chapter under renormalization group flows. In Sec. 5.5 we present our conclusions. In Appendix D.1 we review the form of the phase space path integral for Hamiltonian systems on a general phase space $\mathcal{M}$ equipped with symplectic form ω. In Appendix D.2 we give a brief introduction to the equivariant localization technique for phase space path integrals. Finally, in Appendix D.3 we present the detailed calculations of the regularized determinants which appear in the expression (obtained from the equivariant localization technique) for the partition function of the BTI boundary.

5.2 Perturbative anomalies in bosonic integer quantum Hall states

In this section we study perturbative $U(1)$ anomalies at the boundary of a class of bosonic SPT phases in odd spacetime dimensions which are protected by the symmetry of the group $G = U(1)$. We refer to these phases as bosonic integer quantum Hall (BIQH) states. They are all higher-dimensional generalizations of the $(2+1)$-dimensional BIQH state introduced in Ref. [18]. Upon coupling to a background $U(1)$ gauge field $A = A_\mu dx^\mu$, the boundary of these states exhibits a perturbative $U(1)$ anomaly. For the BIQH phase in $2m-1$ dimensions, the anomaly of the boundary is such that it can be compensated by a bulk Chern-Simons (CS) term

$$S_{CS}[A] = \frac{N_{2m-1}}{(2\pi)^{m-1}m!}\int_X A\wedge(dA)^{m-1} \tag{5.2.1}$$

with the level N_{2m-1} of the CS term quantized in integer multiples of $m!$ (factorial). Here X denotes the $(2m-1)$-dimensional bulk spacetime. We computed this anomaly in Ref. [57] using a NLSM description of the boundary theory of the BIQH state. Specifically, we modeled the boundary using an $O(2m)$ NLSM with Wess-Zumino (WZ) term, with a particular action of the group $U(1)$ on the NLSM field. The field in this model is a $2m$-component unit vector $\mathbf{n} = (n^1,\dots,n^{2m})$, and so the target space of the $O(2m)$ NLSM is the $(2m-1)$-dimensional unit sphere S^{2m-1}.

In this section we first recall the result of Ref. [57], and we also show that the anomaly computed there is well-defined in the sense that it is independent of a certain freedom in the specific form of the terms appearing in the gauged WZ action for the boundary theory. We then consider alternative descriptions of the BIQH state using NLSMs with a general target space $\mathcal{M}$, and we prove that if $\mathcal{M}$ can be obtained from S^{2m-1} by smooth deformations which preserve the $U(1)$ symmetry of the BIQH state, then the anomaly of the NLSM theory with target space $\mathcal{M}$ is *identical* to the anomaly of the $O(2m)$ NLSM theory. Later in the Chapter, in Sec. 5.4, we discuss the expected low energy behavior of the NLSMs discussed in this section.

The results of this section prove that the anomaly computed in Ref. [57] is robust against arbitrary smooth, symmetry-preserving deformations of the NLSM used to describe the boundary of the BIQH state. This is exactly what one hopes for in a model of an SPT phase: smooth, symmetry-preserving deformations of a model of an SPT phase should not affect the ability of that model to capture the universal properties of the SPT phase, provided that the deformations do not take one across a phase boundary. We also note here that in Ref. [57] we gave a more general *gauge invariance* argument for the quantization of the level N_{2m-1} of the CS term describing the bulk response of the BIQH state. That argument also implies that the boundary anomaly is robust and independent of the specific details of any particular model of the boundary of the BIQH state. Therefore, the results of this section could have been anticipated from the gauge invariance argument in Ref. [57]. However, it is also instructive to have an explicit proof of this invariance for the class of NLSM descriptions of the boundary considered here.

5.2.1 Review of $O(2m)$ NLSM calculation of the anomaly

We start by reviewing the calculation of the boundary anomaly of the BIQH state using the $O(2m)$ NLSM description. The boundary of the $(2m-1)$-dimensional BIQH state can be described by an $O(2m)$ NLSM with WZ term. Let X_{bdy} denote the $(2m-2)$-dimensional boundary spacetime. As we discussed above, the NLSM field $\mathbf{n} = (n^1, \dots, n^{2m})$ should be understood as a map $\mathbf{n} : X_{bdy} \to S^{2m-1}$ from the boundary spacetime X_{bdy} to the target space of the NLSM, which is just the unit sphere S^{2m-1} in this case.

The WZ term for the NLSM requires the following ingredients for its construction. First, we need the volume form ω_{2m-1} on S^{2m-1}. In terms of the coordinates n^a, $a = 1, \dots, 2m$, it takes the form

$$\omega_{2m-1} = \sum_{a=1}^{2m} (-1)^{a-1} n^a dn^1 \wedge \cdots \wedge \overline{dn^a} \wedge \cdots dn^{2m} \ , \tag{5.2.2}$$

where the overline means to omit that term from the wedge product. Next, we need an extension $\mathcal{B}$ of the boundary spacetime X_{bdy} such that $\partial\mathcal{B} = X_{bdy}$, where $\partial\mathcal{B}$ denotes the boundary of $\mathcal{B}$. Finally, we need an extension $\tilde{\mathbf{n}}$ of the NLSM field $\mathbf{n}$ into the bulk of $\mathcal{B}$ such that $\tilde{\mathbf{n}}|_{\partial\mathcal{B}} = \mathbf{n}$. The extended field $\tilde{\mathbf{n}}$ should be viewed as a map $\tilde{\mathbf{n}} : \mathcal{B} \to S^{2m-1}$. Then the WZ term for the $O(2m)$ NLSM on the $(2m-2)$-dimensional boundary spacetime X_{bdy} takes the form

$$S_{WZ}[\mathbf{n}] = \frac{2\pi k}{\mathcal{A}_{2m-1}} \int_{\mathcal{B}} \tilde{\mathbf{n}}^* \omega_{2m-1} \ , \tag{5.2.3}$$

where $k \in \mathbb{Z}$ is the level of the WZ term and $\mathcal{A}_{2m-1} = \text{Area}[S^{2m-1}] = \frac{2\pi^m}{(m-1)!}$. Here the notation $\tilde{\mathbf{n}}^*\omega_{2m-1}$ denotes the pullback of the volume form ω_{2m-1} on S^{2m-1} to the extended boundary spacetime $\mathcal{B}$ via the map $\tilde{\mathbf{n}} : \mathcal{B} \to S^{2m-1}$.

The WZ term can be written in a more familiar form if we introduce a system of local coordinates $(s, x^0, \dots, x^{2m-3})$ on $\mathcal{B}$, where $(x^0, \dots, x^{2m-3})$ are a system of local coordinates on X_{bdy}, and where $s \in [0, 1]$ is a coordinate for the extra direction in $\mathcal{B}$. We choose boundary conditions on the extended field configuration such that $\tilde{\mathbf{n}}$ is equal to a trivial constant configuration at $s = 0$, and $\tilde{\mathbf{n}} = \mathbf{n}$ at $s = 1$. Hence, the physical boundary spacetime X_{bdy} is located at $s = 1$. In these coordinates the WZ term takes the more explicit form

$$S_{WZ}[\mathbf{n}] = \frac{2\pi k}{\mathcal{A}_{2m-1}} \int_0^1 ds \int d^{2m-2}x \ \epsilon_{a_1 \cdots a_{2m}} \tilde{n}^{a_1} \partial_s \tilde{n}^{a_2} \partial_{x^0} \tilde{n}^{a_3} \cdots \partial_{x^{2m-3}} \tilde{n}^{a_{2m}} \ , \tag{5.2.4}$$

where we sum over all indices which appear once as a subscript and once as a superscript (the standard summation notation). In addition to the WZ term, the action for the $O(2m)$ NLSM also includes a conventional kinetic term

$$S_{kin}[\mathbf{n}] = \frac{1}{2f} \int d^{2m-2}x \ (\partial_\mu \mathbf{n}) \cdot (\partial^\mu \mathbf{n}) \ , \tag{5.2.5}$$

where f is a coupling constant with dimensions of $(mass)^{4-2m}$ (the power is equal to two minus the boundary spacetime dimension).

The action of the $U(1)$ symmetry that protects the BIQH state on the NLSM field is best described by first pairing the components of $\mathbf{n}$ into m "bosons" $b_\ell = n^{2\ell-1} + in^{2\ell}$, $\ell = 1, \dots, m$. Then, for the NLSM model of the BIQH phase, the $U(1)$ symmetry can be defined to act on these bosons as [18, 57]

$$U(1) : b_\ell \to e^{i\xi} b_\ell \ , \ \forall \, \ell \ . \tag{5.2.6}$$

Let us briefly explain the rationale for this choice of the $U(1)$ action. In the NLSM description of bosonic SPT phases from Ref. [35], the information about the symmetry group G is encoded in a homomorphism $\sigma : G \to SO(2m)$ (in the case of unitary symmetries which have trivial action on spacetime). The NLSM equipped with the homomorphism σ will describe a trivial phase if there exists a vector $\mathbf{v}$ such that $\sigma(g)\mathbf{v} = \mathbf{v}$, $\forall \, g \in G$. This is because in this case it is possible to add a "Zeeman" term $\mathbf{n} \cdot \mathbf{v}$ to the NLSM action to drive the NLSM into a trivial direct product state in which $\mathbf{n}$ is parallel or anti-parallel to $\mathbf{v}$ at all points in space. Therefore, we must choose a homomorphism σ where no such vector $\mathbf{v}$ exists if we want our NLSM to describe a nontrivial SPT phase with $U(1)$ symmetry. Mathematically, the problem is to embed $U(1) \cong SO(2)$ inside the maximal torus of $SO(2m)$ in such a way that no vector $\mathbf{v}$ is fixed under the action of $\sigma(g) \ \forall \, g \in U(1)$. The unique solution to this problem[5], modulo trivial permutations of the components n^a in the definition of the bosons b_ℓ, is the one in Eq. (5.2.6).

Next, we couple the NLSM describing the boundary of the BIQH state to a background $U(1)$ gauge field $A = A_\mu dx^\mu$, and attempt to construct an action which is invariant under the gauge transformation

$$\begin{aligned} b_\ell &\to e^{i\xi} b_\ell \ , \ \forall \, \ell \\ A &\to A + d\xi \ , \end{aligned} \tag{5.2.7}$$

where ξ is now a function of the boundary spacetime coordinates. This gauge transformation can be recast in a more geometric form using the vector field $\underline{v} = v^a \frac{\partial}{\partial n^a}$ which generates the action of the $U(1)$ symmetry on S^{2m-1}. Concretely, this means that under an infinitesimal $U(1)$ transformation, the coordinates on S^{2m-1} transform as

$$n^a \to n^a + \xi v^a \ . \tag{5.2.8}$$

[5] More precisely, this is the unique solution if we demand that the fundamental particles in the model carry unit electric charge.

For the $U(1)$ symmetry action defined in Eq. (5.2.6), the vector field $\underline{v}$ takes the form

$$\underline{v} = \sum_{\ell=1}^{m} \left(-n^{2\ell} \frac{\partial}{\partial n^{2\ell-1}} + n^{2\ell-1} \frac{\partial}{\partial n^{2\ell}} \right) \ . \tag{5.2.9}$$

This transformation of the coordinates also induces a transformation for general p-forms β on S^{2m-1},

$$\beta \to \beta + \mathcal{L}_{\xi \underline{v}} \beta \ , \tag{5.2.10}$$

where $\mathcal{L}_{\underline{v}} = d i_{\underline{v}} + i_{\underline{v}} d$ is the Lie derivative (acting on differential forms) along $\underline{v}$, and $i_{\underline{v}}$ is the interior multiplication by $\underline{v}$ (d is the ordinary exterior derivative).

To simplify the presentation of the gauged WZ action it is best to work with a more compact notation. Let us define the normalized volume form $\alpha^{(2m-1)} = \frac{\omega_{2m-1}}{\mathcal{A}_{2m-1}}$ so that the WZ term can be written as

$$S_{WZ}[\mathbf{n}] = 2\pi k \int_{\mathcal{B}} \tilde{\mathbf{n}}^* \alpha^{(2m-1)} \ . \tag{5.2.11}$$

The derivation of the gauged WZ action is somewhat technical, and so we refer the reader to Ref. [57] for details. In Ref. [57] we showed that the gauged WZ action for the $O(2m)$ NLSM takes the form

$$S_{WZ,gauged}[\mathbf{n}, A] = S_{WZ}[\mathbf{n}] + 2\pi k \sum_{r=1}^{m-1} \int_{X_{bdy}} A \wedge F^{r-1} \wedge \mathbf{n}^* \alpha^{(2m-1-2r)} \ , \tag{5.2.12}$$

where the $\alpha^{(2m-1-2r)}$ are a set of differential forms on S^{2m-1} of degree $2m-1-2r$, $r = 1, \dots, m-1$, which have a form that we now discuss.

First, for each $\ell = 1, \dots, m$, we define one-forms $\mathcal{J}_\ell$ and two-forms $\mathcal{K}_\ell$ on S^{2m-1} by

$$\mathcal{J}_\ell = n_{2\ell-1} dn_{2\ell} - n_{2\ell} dn_{2\ell-1} \tag{5.2.13a}$$

$$\mathcal{K}_\ell = dn_{2\ell-1} \wedge dn_{2\ell} \ . \tag{5.2.13b}$$

Then, for each $r = 0, \dots, m-1$, we define the forms $\Omega^{(r)}$ by

$$\Omega^{(r)} = \sum_{\ell_1, \dots, \ell_{m-r}=1}^{m} \mathcal{J}_{\ell_1} \wedge \mathcal{K}_{\ell_2} \wedge \dots \wedge \mathcal{K}_{\ell_{m-r}} \ . \tag{5.2.14}$$

In particular, $\Omega^{(r)}$ is a form of degree $2m-1-2r$ and the volume form can be expressed in terms of $\Omega^{(0)}$ as

$\omega_{2m-1} = \frac{1}{(m-1)!}\Omega^{(0)}$. One can show that these forms obey the relation

$$i_{\underline{v}}\Omega^{(r)} = \frac{1}{2}d\Omega^{(r+1)} \ , \tag{5.2.15}$$

and this relation allows for the construction of the gauged WZ action. In terms of these forms, the forms $\alpha^{(2m-1-2r)}$ appearing in the gauged WZ action are given by

$$\alpha^{(2m-1-2r)} = \frac{1}{\mathcal{A}_{2m-1}}\frac{1}{(m-1)!}\frac{1}{2^r}\Omega^{(r)} \ . \tag{5.2.16}$$

This collection of forms obeys the set of equations

$$i_{\underline{v}}\alpha^{(2m-1-2r)} = d\alpha^{(2m-1-2r-2)} \ , \ r = 0, \dots, m-2 \ , \tag{5.2.17}$$

and

$$i_{\underline{v}}\alpha^{(1)} = \frac{1}{\mathcal{A}_{2m-1}}\frac{1}{(m-1)!}\frac{1}{2^{m-1}} \ . \tag{5.2.18}$$

Since $\mathcal{A}_{2m-1} = \frac{2\pi^m}{(m-1)!}$, we can rewrite the equations satisfied by the $\alpha^{2m-1-2r}$ as

$$i_{\underline{v}}\alpha^{(2m-1-2r)} = d\alpha^{(2m-1-2r-2)} \ , \ r = 0, \dots, m-2 \ , \tag{5.2.19a}$$

$$i_{\underline{v}}\alpha^{(1)} = \frac{1}{(2\pi)^m} \ . \tag{5.2.19b}$$

Under a $U(1)$ gauge transformation $b_\ell \to e^{i\xi}b_\ell$, $A \to A + d\xi$, the gauged WZ action for the $O(2m)$ NLSM transforms as

$$\delta_\xi S_{WZ,gauged}[\mathbf{n}, A] = k\int_{X_{bdy}} \xi\left(\frac{F}{2\pi}\right)^{m-1} \ . \tag{5.2.20}$$

In the $O(2m)$ NLSM description of the boundary of the BIQH state, this anomaly of the gauged WZ term implies that the topological electromagnetic response of the bulk of the BIQH state is described by a CS term with level $N_{2m-1} = -(m!)k$, i.e., the level must be an integer multiple of $m!$. By inspecting the individual terms in the gauged WZ action, one can see that the anomaly in Eq. (5.2.20) is completely determined by the value of $i_{\underline{v}}\alpha^{(1)}$ as shown in Eqs. (5.2.19). This is because Eq. (5.2.19a) guarantees that the transformation of the form $\alpha^{(2m-1-2r)}$ in the r^{th} term in Eq. (5.2.12) is canceled by the transformation of the gauge field A in the $(r+1)^{th}$ term. This means that the final anomaly only depends on the transformation of $\alpha^{(1)}$ in the $(m-1)^{th}$ term (i.e., the last term). It turns out that the equations which define the form $\alpha^{(1)}$ do not have a unique solution, and in the computation above we have chosen a particular solution. We now show that although there is an ambiguity in the choice of solution for $\alpha^{(1)}$, the anomaly of the gauged action is not affected by this ambiguity.

5.2.2 Uniqueness of the anomaly

In the previous subsection we showed that the anomaly of the $O(2m)$ NLSM with WZ term is completely determined by the one-form $\alpha^{(1)}$ which appears in the final term of the gauged WZ action, and we also mentioned that $\alpha^{(1)}$ is not unique. If we are to ascribe any physical meaning to the anomaly computed in the last subsection, then we need to make sure that the anomaly is not affected by the ambiguity in the choice of the form $\alpha^{(1)}$. In this section we prove that the anomaly is well-defined even though the choice of $\alpha^{(1)}$ is not unique.

We start by precisely characterizing the ambiguity in the choice of the one-form $\alpha^{(1)}$. According to Eqs. (5.2.19), this form should satisfy the equation

$$i_{\underline{v}}\alpha^{(3)} = d\alpha^{(1)} \ . \tag{5.2.21}$$

However, for a given three-form $\alpha^{(3)}$, the solutions to this equation for $\alpha^{(1)}$ are not unique. To see this, let us fix a choice of $\alpha^{(3)}$ (and also $\alpha^{(5)}, \dots, \alpha^{(2m-3)}$) and suppose that we have two solutions $\alpha^{(1)}$ and $\tilde{\alpha}^{(1)}$ to Eq. (5.2.21). If we subtract the equation for $\alpha^{(1)}$ from the equation for $\tilde{\alpha}^{(1)}$ then we find that these two forms are related by the equation

$$d(\tilde{\alpha}^{(1)} - \alpha^{(1)}) = 0 \ , \tag{5.2.22}$$

i.e., the difference $\tilde{\alpha}^{(1)} - \alpha^{(1)}$ is a closed form on S^{2m-1}. However, on the sphere S^{2m-1} all closed one-forms are also exact[6], which means that we have

$$\tilde{\alpha}^{(1)} - \alpha^{(1)} = d\gamma^{(0)} \tag{5.2.23}$$

for some function $\gamma^{(0)}$ on S^{2m-1}.

We now want to understand the possible dependence of the anomaly on the function $\gamma^{(0)}$ which parametrizes the ambiguity in the solution for $\alpha^{(1)}$. Therefore we should compare the gauged WZ action constructed using $\alpha^{(1)}$ with the gauged WZ action constructed using $\tilde{\alpha}^{(1)}$ (but keeping all other terms in the action the same). Let $S_{WZ,gauged}[\mathbf{n}, A]$ be the gauged WZ action constructed using the form $\alpha^{(1)}$, and let $\tilde{S}_{WZ,gauged}[\mathbf{n}, A]$ be the gauged WZ action constructed from the form $\tilde{\alpha}^{(1)}$. These actions differ by a single term

$$\begin{aligned} \tilde{S}_{WZ,gauged}[\mathbf{n}, A] - S_{WZ,gauged}[\mathbf{n}, A] &= 2\pi k \int_{X_{bdy}} A \wedge F^{m-2} \wedge \mathbf{n}^* d\gamma^{(0)} \\ &= 2\pi k \int_{X_{bdy}} \mathbf{n}^*\gamma^{(0)} \ F^{m-1} \ , \end{aligned} \tag{5.2.24}$$

where we rearranged the forms and performed an integration by parts to derive the second equality. Under a gauge

[6] On S^{2m-1} the de Rham cohomology groups $H^r_{dR}(S^{2m-1})$ are trivial for $r = 1, \dots, 2m-2$.

transformation this difference transforms as

$$\delta_\xi \tilde{S}_{WZ,gauged}[\mathbf{n}, A] - \delta_\xi S_{WZ,gauged}[\mathbf{n}, A] = 2\pi k \int_{X_{bdy}} \mathbf{n}^*(\mathcal{L}_{\xi \underline{v}}\gamma^{(0)})\ F^{m-1}\ . \tag{5.2.25}$$

However, since $\gamma^{(0)}$ is a *function*, we have

$$\begin{aligned}
\mathcal{L}_{\xi\underline{v}}\gamma^{(0)} &= d(\xi i_{\underline{v}}\gamma^{(0)}) + \xi i_{\underline{v}} d\gamma^{(0)} \\
&= \xi i_{\underline{v}} d\gamma^{(0)} \\
&= \xi \mathcal{L}_{\underline{v}}\gamma^{(0)}\ ,
\end{aligned} \tag{5.2.26}$$

where we used the fact that $i_{\underline{v}}\gamma^{(0)} = 0$. Then the difference of gauge transformations reduces to

$$\delta_\xi \tilde{S}_{WZ,gauged}[\mathbf{n}, A] - \delta_\xi S_{WZ,gauged}[\mathbf{n}, A] = 2\pi k \int_{X_{bdy}} \xi\ \mathbf{n}^*(\mathcal{L}_{\underline{v}}\gamma^{(0)})\ F^{m-1}\ . \tag{5.2.27}$$

We can now make the following observation. The gauged action $S_{WZ,gauged}[\mathbf{n}, A]$ constructed using $\alpha^{(1)}$ from Eq. (5.2.16) still possesses *global* $U(1)$ symmetry and, in particular, is invariant under the transformation $b_\ell \to e^{i\xi} b_\ell$ for an infinitesimal *constant* parameter ξ. However, the above considerations show that under the same infinitesimal $U(1)$ transformation, the gauged action $\tilde{S}_{WZ,gauged}[\mathbf{n}, A]$ constructed from $\tilde{\alpha}^{(1)}$ will transform as

$$\delta_\xi \tilde{S}_{WZ,gauged}[\mathbf{n}, A] = 2\pi k \xi \int_{X_{bdy}} \mathbf{n}^*(\mathcal{L}_{\underline{v}}\gamma^{(0)})\ F^{m-1}\ . \tag{5.2.28}$$

Now even if the gauged WZ action cannot be made to be invariant under $U(1)$ gauge transformations, we should still require it to be invariant under *global* $U(1)$ transformations. Therefore we must demand that for any alternative solution $\tilde{\alpha}^{(1)}$ to Eq. (5.2.21), the function $\gamma^{(0)}$ relating this form to $\alpha^{(1)}$ from Eq. (5.2.16) should satisfy

$$\mathcal{L}_{\underline{v}}\gamma^{(0)} = 0\ , \tag{5.2.29}$$

i.e., this function should be invariant under the action of the $U(1)$ symmetry on S^{2m-1}. Then, since we have the relation $\mathcal{L}_{\xi\underline{v}}\gamma^{(0)} = \xi\mathcal{L}_{\underline{v}}\gamma^{(0)}$ for any function $\gamma^{(0)}$ and any spacetime-dependent ξ, we immediately find that the anomaly of the gauged WZ action is not sensitive to the ambiguity in the choice of $\alpha^{(1)}$. In other words, the requirement that the gauged WZ action should still possess *global* $U(1)$ symmetry is enough to ensure that the anomaly of the gauged action is well-defined and independent of the ambiguity in the choice of $\alpha^{(1)}$.

5.2.3 Deforming the target space

Now that we know that the anomaly in Eq. (5.2.20) is well-defined, we can move on and study how deformations of the target space of the NLSM might affect the anomaly. Recall that we previously derived this anomaly using the $O(2m)$ NLSM with target space S^{2m-1}. In this subsection we show that this anomaly is not affected by arbitrary smooth, symmetry-preserving deformations of the target space of the NLSM. The notion of a smooth, symmetry-preserving deformation of the target space can be formulated precisely in terms of diffeomorphisms which are *equivariant* with respect to the symmetry action, as we discuss below.

In the NLSM description of the BIQH state the target space S^{2m-1} of the $O(2m)$ NLSM is equipped with an action of the group $U(1)$. For any $g \in U(1)$ let us write $g \cdot \mathbf{n}$ to denote the image of the point $\mathbf{n} \in S^{2m-1}$ under the action of the group element g. As we discussed above, the $U(1)$ action on S^{2m-1} is generated by the vector field $\underline{v}$ in the sense that $n^a \to n^a + \xi v^a$ under an infinitesimal $U(1)$ transformation parametrized by ξ. Now suppose that $\mathcal{M}$ is another $(2m-1)$-dimensional manifold with the following properties.

(1) There is a $U(1)$ action on $\mathcal{M}$ generated by a vector field $\underline{w}$.

(2) There exists a Riemannian metric on $\mathcal{M}$ for which $\underline{w}$ is a Killing vector.

(3) There exists a diffeomorphism $f : \mathcal{M} \to S^{2m-1}$ which is *equivariant* with respect to the $U(1)$ action, i.e.,

$$g \cdot f(\mathbf{m}) = f(g \cdot \mathbf{m}) \ , \ \forall\, \mathbf{m} \in \mathcal{M} \ , \ \forall\, g \in U(1) \ . \tag{5.2.30}$$

Intuitively, these properties imply that the manifold $\mathcal{M}$ also has a $U(1)$ symmetry, and that it can be reached from S^{2m-1} (or vice-versa) by smooth deformations which respect the $U(1)$ symmetry. We now show that for any such manifold $\mathcal{M}$ the NLSM with target space $\mathcal{M}$, WZ term at level k, and $U(1)$ action generated by $\underline{w}$ possesses the exact same perturbative $U(1)$ anomaly as the $O(2m)$ NLSM with WZ term at level k.

Before presenting the proof, we first discuss some consequences of the three properties of the map f. First, properties (1) and (2) together imply that we can construct a WZ term for the NLSM with target space $\mathcal{M}$ with the property that the WZ term is invariant under the $U(1)$ transformation generated by $\underline{w}$ (we construct the WZ term using the volume form on $\mathcal{M}$ determined by its $U(1)$-symmetric Riemannian metric). Next, the first part of property (3), namely the fact that $f : \mathcal{M} \to S^{2m-1}$ is a diffeomorphism, implies that the de Rham cohomology groups of $\mathcal{M}$ and S^{2m-1} are identical. In addition, the fact that f is a diffeomorphism implies that the *degree* of f, defined via the equation

$$\begin{aligned}\frac{1}{\mathcal{A}_{2m-1}}\int_{\mathcal{M}} f^*\omega_{2m-1} &= \deg[f]\frac{1}{\mathcal{A}_{2m-1}}\int_{S^{2m-1}}\omega_{2m-1} \\ &= \deg[f]\ ,\end{aligned} \tag{5.2.31}$$

is equal to plus or minus one, $\deg[f] = \pm 1$ (see Ch. VI of Ref. [212] for the definition of the degree of a smooth map). Intuitively this means that the map f "wraps" $\mathcal{M}$ around S^{2m-1} only once. This has to be the case since f is injective (f is invertible so it is both injective and surjective). In what follows we assume $\deg[f] = 1$ so that f is orientation-preserving. This then implies that

$$f^*\left(\frac{\omega_{2m-1}}{\mathcal{A}_{2m-1}}\right) = \frac{\omega_{\mathcal{M}}}{\mathcal{A}_{\mathcal{M}}}\ , \tag{5.2.32}$$

where $\omega_{\mathcal{M}}$ is the volume form on $\mathcal{M}$ determined by its Riemannian metric, and $\mathcal{A}_{\mathcal{M}} = \int_{\mathcal{M}}\omega_{\mathcal{M}}$ is the area of $\mathcal{M}$.

Next, properties (1) and (3) together imply that

$$\underline{v} = f_*\underline{w}\ , \tag{5.2.33}$$

i.e., the vector field $\underline{v}$ which generates the $U(1)$ action on S^{2m-1} is equal to the pushforward, via the map f, of the vector field $\underline{w}$ that generates the $U(1)$ action on $\mathcal{M}$. This can be verified by expanding out both sides of Eq. (5.2.30) for an element $g \in U(1)$ which is close to the identity. This property implies the following relation, which is central to the proof in this section. If α is a differential form on S^{2m-1}, then we have

$$i_{\underline{w}}(f^*\alpha) = f^*(i_{\underline{v}}\alpha)\ . \tag{5.2.34}$$

This relation implies that the action of interior multiplication commutes with the action of taking the pullback, provided that we use $i_{\underline{w}}$ when acting on forms on $\mathcal{M}$ and $i_{\underline{v}}$ when acting on forms on S^{2m-1}. Again, this relation holds because under our assumptions the vector field $\underline{v}$ is equal to the pushforward of $\underline{w}$ by the map f.

Now let us consider an alternative description of the boundary of a BIQH state in terms of a NLSM with target space $\mathcal{M}$, where $\mathcal{M}$ satisfies the three properties stated above. The field in this NLSM theory, which we denote by $\mathbf{m}$, is a map from the boundary spacetime to the manifold $\mathcal{M}$, $\mathbf{m} : X_{bdy} \to \mathcal{M}$. We also assume that the transformation of the NLSM field $\mathbf{m}$ under the $U(1)$ symmetry of the BIQH state is determined by the $U(1)$ action on $\mathcal{M}$ generated by the vector field $\underline{w}$. For example, under an infinitesimal $U(1)$ transformation parametrized by ξ we have $m^a \to m^a + \xi w^a$, $\forall\ a$. The WZ term for this NLSM is constructed in the same way as for the NLSM with target space S^{2m-1}. We start with a volume form $\omega_{\mathcal{M}}$ on $\mathcal{M}$ which we assume is obtained from a $U(1)$-symmetric Riemannian metric on $\mathcal{M}$[7] (which exists by our assumption (2) above). We denote the normalized volume form on $\mathcal{M}$ by $\beta^{(2m-1)} = \frac{\omega_{\mathcal{M}}}{\mathcal{A}_{\mathcal{M}}}$, where $\mathcal{A}_{\mathcal{M}} = \int_{\mathcal{M}} \omega_{\mathcal{M}}$. We also need an extension $\tilde{\mathbf{m}}$ of the NLSM field $\mathbf{m}$ into the extended boundary spacetime $\mathcal{B}$ such that $\tilde{\mathbf{m}}|_{\partial\mathcal{B}} = \mathbf{m}$. In terms of these quantities, the WZ term for the NLSM with target space $\mathcal{M}$ can be written in the compact form

$$S_{WZ}[\mathbf{m}] = 2\pi k \int_{\mathcal{B}} \tilde{\mathbf{m}}^* \beta^{(2m-1)} \ . \tag{5.2.35}$$

We can now attempt to couple $S_{WZ}[\mathbf{m}]$ to the gauge field A and study the perturbative anomaly of the gauged action. We find that the gauged WZ term for the NLSM with target space $\mathcal{M}$ takes the form

$$S_{WZ,gauged}[\mathbf{m}, A] = S_{WZ}[\mathbf{m}] + 2\pi k \sum_{r=1}^{m-1} \int_{X_{bdy}} A \wedge F^{r-1} \wedge \mathbf{m}^* \beta^{(2m-1-2r)} \ , \tag{5.2.36}$$

where the forms $\beta^{(2m-1-2r)}$ on $\mathcal{M}$ are obtained by pulling back the forms $\alpha^{(2m-1-2r)}$ on S^{2m-1} which appear in the gauged WZ action for the $O(2m)$ NLSM,

$$\beta^{(2m-1-2r)} = f^* \alpha^{(2m-1-2r)} \ . \tag{5.2.37}$$

The explicit form of $\alpha^{(2m-1-2r)}$ was given above in Eq. (5.2.16). Using Eq. (5.2.34) and the fact that the pullback operation commutes with the exterior derivative, we find that the forms $\beta^{(2m-1-2r)}$ for $r = 0, 1, \dots, m-1$, obey the set of equations

$$i_{\underline{w}} \beta^{(2m-1-2r)} = d\beta^{(2m-1-2r-2)} \ , \ r = 0, \dots, m-2 \ , \tag{5.2.38a}$$

$$i_{\underline{w}} \beta^{(1)} = \frac{1}{(2\pi)^m} \ . \tag{5.2.38b}$$

These equations are identical to Eqs. (5.2.19) but with $\underline{v}$ replaced by $\underline{w}$ and $\alpha^{(2m-1-2r)}$ replaced by $\beta^{(2m-1-2r)}$. The form of these equations implies that the NLSM theory with target space $\mathcal{M}$ has the exact same perturbative $U(1)$ anomaly as the $O(2m)$ NLSM with target space S^{2m-1}. In addition, our argument for the uniqueness of the anomaly from the previous subsection also applies to the theory with target space $\mathcal{M}$. This follows from the fact that the de Rham cohomology groups of $\mathcal{M}$ are identical to those of S^{2m-1} as a consequence of our assumption (3). Therefore we have shown that the perturbative $U(1)$ anomaly at the boundary of the BIQH state is robust against arbitrary smooth,

[7] Although we did not discuss it explicitly, our earlier construction of the WZ term for the $O(2m)$ NLSM also required a $U(1)$-symmetric Riemannian metric for S^{2m-1}. In particular, the volume form ω_{2m-1} is the volume form on S^{2m-1} which is obtained from the natural Riemannian metric on S^{2m-1} induced by the embedding of S^{2m-1} in $\mathbb{R}^{2m}$. The $U(1)$ symmetry of this metric then implied that the $O(2m)$ NLSM with WZ term possessed a global $U(1)$ symmetry.

symmetry-preserving deformations of the target space of the NLSM used to describe the BIQH state.

5.3 Global anomalies in Bosonic Topological Insulator states

In this section we study global anomalies at the boundary of a class of bosonic SPT phases which exist in even spacetime dimensions and are protected by the symmetry of the group $G = U(1) \rtimes \mathbb{Z}_2$. We refer to these phases as bosonic topological insulator (BTI) phases. They are generalizations to all even-dimensional spacetimes of the BTI phase introduced in Ref. [19]. Note also that the system of bosons studied in Ref. [213] can be considered to be an example of a $(1+1)$-dimensional BTI state according to our definition. In all cases the $U(1)$ symmetry represents a physical charge conservation symmetry, however, the character of the $\mathbb{Z}_2$ symmetry depends on the specific dimension of spacetime. Let the bulk spacetime dimension be $2m$ for a positive integer m. Then for m odd the $\mathbb{Z}_2$ symmetry is a unitary charge-conjugation symmetry, while for m even the $\mathbb{Z}_2$ symmetry is an anti-unitary time-reversal symmetry.

In Ref. [57] we argued that the boundary theory of the $2m$-dimensional BTI state exhibits a bosonic analogue of the parity anomaly of a Dirac fermion in odd dimensions. Our argument was based on the form of the gauged WZ action in an $O(2m+1)$ NLSM description of the boundary of these phases. Specifically, we showed that if the NLSM field on the boundary of the BTI condensed in such a way as to break the $\mathbb{Z}_2$ symmetry but preserve the $U(1)$ symmetry of the BTI phase, then the boundary would exhibit a BIQH response with *half-quantized* CS coefficient $N_{2m-1} = \frac{m!}{2}$. We then argued by analogy with the free fermion topological insulator [7, 8] that this half-quantized BIQH response indicated that the boundary of the BTI phase displays a bosonic analogue of the parity anomaly.

In this section we make this reasoning precise in the special case of the BTI state in $1+1$ spacetime dimensions. In this case we are able to compute the boundary partition function exactly, and the global anomaly can be seen clearly from our exact result. We start by reviewing the form of the $O(3)$ NLSM action which describes the $(0+1)$-dimensional boundary of this BTI state, including the form of the gauged WZ action which describes the boundary theory coupled to the external gauge field A [57]. We then explicitly compute the boundary partition function and show that it cannot retain both the $\mathbb{Z}_2$ symmetry of the BTI *and* large $U(1)$ gauge invariance, i.e., the boundary theory possesses a *global anomaly* in the $\mathbb{Z}_2$ symmetry of the BTI state. We then consider arbitrary smooth, symmetry-preserving deformations of the target space of the NLSM used to describe the BTI, and we use the powerful *equivariant localization* (EL) technique to show that the boundary partition function and the global anomaly are robust against such deformations of the model. We also note here that the global anomaly computed in this section is very similar to the global anomaly computed in Ref. [76] for a single Dirac fermion in $(0+1)$-dimensions with $U(1)$ symmetry and unitary $\mathbb{Z}_2$ charge-conjugation symmetry.

5.3.1 The BTI state in $1+1$ dimensions and its O(3) NLSM description

The BTI state in $1+1$ dimensions is an SPT phase of bosons with symmetry group $G = U(1) \rtimes \mathbb{Z}_2$, where $U(1)$ represents charge conservation and $\mathbb{Z}_2$ is a unitary charge-conjugation (or particle-hole) symmetry. The semi-direct product "$\rtimes$" indicates that the $U(1)$ and $\mathbb{Z}_2$ symmetries do not commute with each other. The physical signature of the BTI state is that a fractional charge of $\pm\frac{1}{2}$ (in units of the boson charge) is bound at an interface between the BTI state and the vacuum (or a trivial state). One possible model for the bulk of the BTI state is an $O(3)$ NLSM with theta term and coefficient $\theta = 2\pi k$, $k \in \mathbb{Z}$ [35]. The boundary of the BTI state is then described by the same NLSM but with a WZ term at level k. In $1+1$ dimensions SPT phases with the symmetry group $G = U(1) \rtimes \mathbb{Z}_2$ have a $\mathbb{Z}_2$ classification, meaning that there is only a single nontrivial phase [14, 35]. This single nontrivial phase is the BTI state. Within the NLSM description, the NLSM for any odd k represents the nontrivial BTI state, while the model for any even k represents the trivial state.

In the $O(3)$ NLSM description the field is a unit vector field $\mathbf{n}$ with components n^a, $a = 1, 2, 3$. The target space of the $O(3)$ NLSM is the unit two-sphere S^2. As in Sec. 5.2, the action of the symmetry group $G = U(1) \rtimes \mathbb{Z}_2$ of the BTI on the NLSM field is best expressed by first combining n^1 and n^2 into the "boson" field $b = n^1 + in^2$. Then for the BTI state, the action of G on the NLSM field is given by (see Sec. IV.D of Ref. [35])

$$U(1) : b \to e^{i\xi} b \,, \tag{5.3.1}$$

and

$$\mathbb{Z}_2 : b \quad \to \quad b^* \tag{5.3.2a}$$

$$n^3 \quad \to \quad -n^3 \,. \tag{5.3.2b}$$

Since the $\mathbb{Z}_2$ symmetry is unitary, the transformation $b \to b^*$ is equivalent to $n^1 \to n^1$, $n^2 \to -n^2$. We can interpret b as the field which annihilates a boson of charge 1, and n^3 can be interpreted as the deviation of the boson density from a non-zero constant value.

The theta term and the WZ term for the $O(3)$ NLSM are both expressed in terms of the volume form ω_2 on S^2,

$$\omega_2 = n^1 dn^2 \wedge dn^3 - n^2 dn^1 \wedge dn^3 + n^3 dn^1 \wedge dn^2 \,. \tag{5.3.3}$$

In what follows we use $\mathcal{A}_2 = 4\pi$ to denote the surface area of S^2 (and $\int_{S^2} \omega_2 = \mathcal{A}_2$). In this Chapter we are only interested in the boundary theory of the BTI, and so we focus our attention on the WZ term. The boundary theory lives in one spacetime dimension. To make our discussion as precise as possible, we take the time coordinate (the only coordinate here) to lie in the interval $t \in [0, T)$, and we impose periodic boundary conditions in the time direction. This makes our one-dimensional spacetime into a circle of circumference T. Let us denote the one-dimensional spacetime by S^1_T (the circle of circumference T). Constructing the WZ term requires extending the spacetime into a two-dimensional spacetime $\mathcal{B}$ such that $\partial\mathcal{B} = S^1_T$. We use $\tilde{\mathbf{n}}$ to denote the extension of the NLSM field $\mathbf{n}$ into the bulk of $\mathcal{B}$, and we require that $\tilde{\mathbf{n}}|_{\partial\mathcal{B}} = \mathbf{n}$. Using $\mathcal{B}$ and the extension $\tilde{\mathbf{n}}$ of $\mathbf{n}$, the WZ term takes the form

$$S_{WZ}[\mathbf{n}] = \frac{2\pi k}{\mathcal{A}_2}\int_{\mathcal{B}} \tilde{\mathbf{n}}^*\omega_2 \ , \tag{5.3.4}$$

where $\tilde{\mathbf{n}}^*\omega_2$ denotes the pullback of ω_2 to $\mathcal{B}$ via the map $\tilde{\mathbf{n}} : \mathcal{B} \to S^2$, and k is the level of the WZ term (the same integer k determines the coefficient $\theta = 2\pi k$ of the theta term describing the bulk of the SPT phase).

The complete $O(3)$ NLSM action describing the boundary of the BTI takes the form

$$S_{bdy}[\mathbf{n}] = \int_0^T dt\ \frac{1}{2f_{bdy}}\left[(\partial^t b)^*(\partial_t b) + (\partial^t n^3)(\partial_t n^3)\right] + S_{WZ}[\mathbf{n}] \ , \tag{5.3.5}$$

where f_{bdy} is a boundary coupling constant and $\partial^t = \partial_t$ for our choice of the signature of the spacetime metric (we use a "mostly minus" Minkowski metric). We can now consider coupling the boundary theory to an external $U(1)$ gauge field $A = A_t dt$. In Ref. [57], we showed that the properly gauged boundary action has the form

$$S_{bdy,gauged}[\mathbf{n}, A] = \int_0^T dt\ \frac{1}{2f_{bdy}}\left[(D^t b)^*(D_t b) + (\partial^t n^3)(\partial_t n^3)\right] + S_{WZ,gauged}[\mathbf{n}, A] \ , \tag{5.3.6}$$

where

$$S_{WZ,gauged}[\mathbf{n}, A] = S_{WZ}[\mathbf{n}] + \frac{2\pi k}{\mathcal{A}_2}\int_0^T dt\ n^3 A_t \ , \tag{5.3.7}$$

and $D_t = \partial_t - iA_t$ ($\partial_t = \partial^t$, $A_t = A^t$, etc., for our choice of signature). The action for the fully gauged boundary theory is invariant under $U(1)$ gauge transformations

$$b \ \to \ e^{i\xi} b \tag{5.3.8a}$$

$$A \ \to \ A + d\xi \ , \tag{5.3.8b}$$

and $\mathbb{Z}_2$ transformations

$$b \ \to \ b^* \tag{5.3.9a}$$

$$n^3 \ \to \ -n^3 \tag{5.3.9b}$$

$$A \ \to \ -A \ . \tag{5.3.9c}$$

5.3.2 Boundary partition function and global anomaly

We now study the partition function for the gauged boundary theory of the BTI in the topological limit $f_{bdy} \to \infty$. In this limit we keep only the low energy information about the boundary theory, including possible anomalies. The partition function

$$Z[A] = \int [d\mathbf{n}] \, e^{iS_{bdy,gauged}[\mathbf{n},A]} \tag{5.3.10}$$

of the gauged boundary theory can be evaluated very simply in this limit, as we now discuss. First, in the limit $f_{bdy} \to \infty$ the path integral we need to evaluate is

$$Z[A] = \int [d\mathbf{n}] \, e^{iS_{WZ,gauged}[\mathbf{n},A]} \ , \tag{5.3.11}$$

where $S_{WZ,gauged}[\mathbf{n}, A]$ is the gauged WZ action from Eq. (5.3.7). The path integral measure appearing here has the precise definition

$$[d\mathbf{n}] = \prod_{t\in[0,T)} \omega_2(t) \ , \tag{5.3.12}$$

where $\omega_2(t)$ denotes the volume form for a copy of S^2 located at the point t in spacetime, and we integrate over all field configurations with periodic boundary conditions in time.

We can also use a gauge transformation to simplify the form of the coupling to the gauge field A. In one spacetime dimension the gauge field one-form $A = A_t dt$ has only one component. Since our spacetime is a circle, which has first cohomology group $H^1(S^1, \mathbb{R}) = \mathbb{R}$, we can decompose a generic A_t as

$$A_t = \overline{A}_t + \partial_t \lambda \ , \tag{5.3.13}$$

where

$$\overline{A}_t := \frac{1}{T} \int_0^T dt \ A_t \tag{5.3.14}$$

represents the nontrivial part of A, and $\partial_t \lambda$ represents the exact part of A (here λ is some function of t). The exact part of A can be removed from the action via a small $U(1)$ gauge transformation, which are those gauge transformations $A \to A+\partial_t\xi$ with the function ξ satisfying $\xi(0) = \xi(T)$. Large $U(1)$ gauge transformations are those transformations which send $\overline{A}_t \to \overline{A}_t + \frac{2\pi n}{T}$, for any $n \in \mathbb{Z}$, and they will play an important role in the discussion of the global anomaly in this theory later in this section. The upshot of all of this is that we can replace the coupling to A_t in the gauged WZ action with a coupling to the *constant* gauge field $\overline{A}_t$.

We now move on to the calculation of the partition function $Z[A]$. We compute the partition function by observing that it is identical to the phase space path integral for a spin of magnitude $J = \frac{k}{2}$ (or $\frac{|k|}{2}$ for negative k) in a constant magnetic field B pointing in the 3-direction, with the magnitude of the magnetic field given in terms of the gauge field A by $B = -\overline{A}_t$. To prove this we now briefly review the form of the phase space path integral for spin. At this point we recommend that the reader skim through Appendix D.1 where we review the phase space path integral expression for the partition function of a quantum mechanical system obtained by quantizing a general classical system defined on a phase space $\mathcal{M}$ equipped with a symplectic form ω and Hamiltonian function H.

The classical mechanics of a spin $J = \frac{1}{2}, 1, \frac{3}{2}, \dots,$ is described by a phase space $\mathcal{M} = S^2$ equipped with the symplectic form $\omega = J\omega_2$, where ω_2 is the volume form on S^2 from Eq. (5.3.3). It is convenient to work in spherical coordinates (ϕ, θ) on S^2. In this system of coordinates the components of the NLSM field $\mathbf{n}$ take the form

$$n^1 = \sin(\theta)\cos(\phi) \tag{5.3.15a}$$

$$n^2 = \sin(\theta)\sin(\phi) \tag{5.3.15b}$$

$$n^3 = \cos(\theta)\ , \tag{5.3.15c}$$

and we have

$$\omega = J\sin(\theta) d\theta \wedge d\phi\ . \tag{5.3.16}$$

Using the definition Eq. (D.1.4) for the Poisson bracket one can check that

$$\{n^a, n^b\} = \frac{1}{J}\sum_c \epsilon^{abc} n^c\ , \tag{5.3.17}$$

so that the spin components S^a are given in terms of n^a by

$$S^a = J n^a\ . \tag{5.3.18}$$

The spin components then obey the Poisson algebra

$$\{S^a, S^b\} = \sum_c \epsilon^{abc} S^c\ . \tag{5.3.19}$$

We can now see that replacing the Poisson bracket with a commutator according to the rule $\{\cdot,\cdot\} \to -i[\cdot,\cdot]$ will give the usual commutation relations for spin in quantum mechanics.

Now let us assume that the dynamics of the spin system are specified by a Hamiltonian H. Then the phase space path integral representing the partition function $\text{tr}_J[e^{-iHT}]$, where the trace is taken in the spin J representation of

$SU(2)$, takes the form

$$\mathrm{tr}_J[e^{-iHT}] = \int [d\phi d\theta] \left[\prod_{t\in[0,T)} J\sin(\theta(t)) \right] e^{iS[\phi,\theta]} \ . \tag{5.3.20}$$

where

$$S[\phi,\theta] = \int_0^T dt \ \left[\vartheta_\phi \partial_t \phi + \vartheta_\theta \partial_t \theta - H(\theta,\phi) \right] \ . \tag{5.3.21}$$

Here ϑ_ϕ and ϑ_θ are the components of the symplectic potential ϑ, which is defined locally on the phase space by the relation $\omega = d\vartheta$ (Eq. (D.1.8) in Appendix D.1). Then, since

$$(\vartheta_\phi \partial_t \phi + \vartheta_\theta \partial_t \theta) dt = \mathbf{n}^* \vartheta \ , \tag{5.3.22}$$

we can rewrite the first term in this action using an extension $\mathcal{B}$ of the spacetime S^1_T and an extension $\tilde{\mathbf{n}}$ of the field configuration (satisfying $\tilde{\mathbf{n}}|_{\partial\mathcal{B}} = \mathbf{n}$). We have

$$\begin{aligned} \int_{S^1_T} \mathbf{n}^* \vartheta &= \int_{\mathcal{B}} \tilde{\mathbf{n}}^* \omega \\ &= J \int_{\mathcal{B}} \tilde{\mathbf{n}}^* \omega_2 \ , \end{aligned} \tag{5.3.23}$$

where the first line follows from Stokes' theorem. If we choose the Hamiltonian to be

$$H = BS^3 = BJn^3 \ , \tag{5.3.24}$$

which is the Hamiltonian for a spin in a constant magnetic field of magnitude B and pointing in the 3-direction, then the action becomes

$$S[\phi,\theta] = J \int_{\mathcal{B}} \tilde{\mathbf{n}}^* \omega_2 - J \int_{S^1_T} n^3 B \ . \tag{5.3.25}$$

We can now compare the path integral for a spin in a magnetic field to our path integral in Eq. (5.3.11) for the partition function of the boundary of the BTI state. Using the fact that $\mathcal{A}_2 = 4\pi$, we find that these path integrals are identical if we make the identifications

$$J = \frac{k}{2} \tag{5.3.26a}$$

$$B = -\overline{A}_t \ . \tag{5.3.26b}$$

More precisely, the path integrals are not identical but differ by the infinite constant factor

$$\prod_{t\in[0,T)} J \,, \tag{5.3.27}$$

but we can give a more careful definition of the path integral measure for the partition function of the gauged boundary theory for the BTI by including this factor. Using all of this information we then find that

$$\begin{aligned} Z[A] &= \mathrm{tr}_{\frac{k}{2}}[e^{iS^3\overline{A}_t T}] \\ &= \sum_{j=-\frac{k}{2}}^{\frac{k}{2}} e^{ij\overline{A}_t T} \\ &= \frac{\sin\left[\frac{\overline{A}_t T}{2}(k+1)\right]}{\sin\left[\frac{\overline{A}_t T}{2}\right]} \,. \end{aligned} \tag{5.3.28}$$

Note that in deriving this formula we assumed that $k > 0$. For $k < 0$ one just needs to replace k with $|k|$. For the discussion below it is useful to decompose the gauge field as $\overline{A}_t = \frac{2\pi\ell}{T} + \overline{a}_t$ for some $\ell \in \mathbb{Z}$ and $\overline{a}_t \in (0, \frac{2\pi}{T})$, and to then rewrite $Z[A]$ in terms of ℓ and $\overline{a}_t$,

$$Z[A] = (-1)^{k\ell}\frac{\sin\left[\frac{\overline{a}_t T}{2}(k+1)\right]}{\sin\left[\frac{\overline{a}_t T}{2}\right]} \,. \tag{5.3.29}$$

It is important to observe that the factor $(-1)^{k\ell}$ is nontrivial for odd k. This minus sign is related to the global anomaly in this theory for odd k, as we now discuss.

For any level k the partition function $Z[A]$ respects the $\mathbb{Z}_2$ symmetry of the BTI state, i.e., we have

$$Z[-A] = Z[A] \,. \tag{5.3.30}$$

However, for odd k the partition function is not invariant under a large $U(1)$ gauge transformation,

$$\overline{A}_t \to \overline{A}_t + \frac{2\pi}{T} \,, \tag{5.3.31}$$

which is equivalent to the transformation $\ell \to \ell + 1$ if we decompose the gauge field as $\overline{A}_t = \frac{2\pi\ell}{T} + \overline{a}_t$. Instead, for odd k the partition function $Z[A]$ changes sign under this transformation. We can try to fix this large gauge invariance issue by modifying the partition function to

$$\tilde{Z}[A] = Z[A]e^{\pm\frac{i}{2}\overline{A}_t T} \,. \tag{5.3.32}$$

This is equivalent to adding the local counterterm $\pm\frac{1}{2}\int_0^T dt\ A_t$ to the original boundary action, which is a $(0+1)$-dimensional Chern-Simons term with fractional level $\pm\frac{1}{2}$. Note, however, that adding this counterterm spoils the invariance of the partition function under the action of the $\mathbb{Z}_2$ symmetry. Therefore we find that although the gauged action $S_{WZ,gauged}[\mathbf{n}, A]$ for the BTI boundary has large $U(1)$ gauge invariance *and* $\mathbb{Z}_2$ symmetry, the partition function $Z[A]$ for the boundary theory only has both of these symmetries when k is even.

This is a classic sign of a *global anomaly* in the $\mathbb{Z}_2$ symmetry: for odd k we can quantize the theory in such a way as to keep either the $\mathbb{Z}_2$ symmetry or large $U(1)$ gauge invariance, but not both. Physically, this anomaly is related to the fact that for odd k the boundary of the BTI has states with half-integer (i.e., fractional) charge. In addition, the fact that the presence or absence of the anomaly depends only on the parity of k (even or odd) is due to the aforementioned $\mathbb{Z}_2$ classification of bosonic SPT phases with $G = U(1) \rtimes \mathbb{Z}_2$ symmetry in $1+1$ dimensions (the theories with odd k all represent the nontrivial BTI state, while the theories with even k all represent the trivial phase). As we discussed above, the anomaly here is very similar to the global anomaly computed in Ref. [76] for a Dirac fermion in $0+1$ dimensions with $U(1)$ and $\mathbb{Z}_2$ symmetry. In addition, a similar anomaly in the (purely bosonic) $(0+1)$-dimensional theory of a particle on a ring was discussed recently in Appendix D of Ref. [214].

5.3.3 Deforming the target space

In the previous subsection we showed that, at least within the $O(3)$ NLSM description, the boundary of the $(1+1)$-dimensional BTI phase exhibits a global anomaly in the $\mathbb{Z}_2$ symmetry of the BTI phase. However, our derivation of the anomaly seemed to rely on the specific geometry of the target space S^2 of the $O(3)$ NLSM. Specifically, our derivation used the fact that the partition function for the BTI boundary was equivalent to a phase space path integral for a spin in a magnetic field. In addition, since $U(1) \rtimes \mathbb{Z}_2$ is a subgroup of $SO(3)$, the anomaly we derived is closely related to the global $SO(3)$ anomaly of the $O(3)$ NLSM with WZ term in $0+1$ dimensions (see, for example, the discussion in Sec. 1.2 of Ref. [215]). Our calculation then shows that the $U(1) \rtimes \mathbb{Z}_2$ subgroup of $SO(3)$ is also anomalous in this theory.

In the rest of this section we show that the boundary anomaly of the BTI state is not affected by any smooth deformation of the target space S^2 of the $O(3)$ NLSM which also preserves the $U(1) \rtimes \mathbb{Z}_2$ symmetry of the BTI phase. In other words, we break the $SO(3)$ symmetry of the model down to $U(1) \rtimes \mathbb{Z}_2$, and we show that the anomaly still exists in these less symmetric theories.

In this subsection we describe the geometry of such deformed target spaces, and then we construct models of the BTI boundary using WZ terms for NLSMs with these deformed target spaces. We also show how to properly gauge these WZ actions. In the next subsection we use the equivariant localization (EL) technique to compute the partition function for these models, and we show that all such models have a partition function which is identical to Eq. (5.3.28).

Thus, we find that the boundary anomaly is completely unaffected by smooth, symmetry-preserving deformations of the target space of the NLSM.

As stated above, we consider descriptions of the BTI using NLSMs with a target space $\mathcal{M}$ that can be obtained from the target space S^2 of the $O(3)$ NLSM by smooth deformations which preserve the $G = U(1) \rtimes \mathbb{Z}_2$ symmetry of the BTI phase. As in Sec. 5.2, we can characterize such spaces $\mathcal{M}$ precisely through the notion of a diffeomorphism which is *equivariant* with respect to the symmetry of the BTI phase. The target space of the $O(3)$ NLSM is S^2, and the NLSM description of the BTI phase includes an action of the group $G = U(1) \rtimes \mathbb{Z}_2$ on S^2. This action was shown explicitly in Eq. (5.3.1) and Eqs. (5.3.2). Let us assume that the manifold $\mathcal{M}$ is also equipped with an action of the group G. Then a diffeomorphism $f : \mathcal{M} \to S^2$ is equivariant with respect to G if

$$f(g \cdot \mathbf{m}) = g \cdot f(\mathbf{m}),\ \forall g \in G\ ,\ \forall \mathbf{m} \in \mathcal{M}\ . \tag{5.3.33}$$

This is the correct mathematical notion corresponding to the intuitive idea of a manifold which can be obtained from S^2 by smooth, symmetry-preserving deformations.

The spaces $\mathcal{M}$ which are related to S^2 in this way can be realized as surfaces of revolution in $\mathbb{R}^3$ which are symmetric under rotation about the z-axis (this guarantees $U(1)$ symmetry), and which are also invariant under reflection $z \to -z$ through the x-y plane[8]. The latter condition guarantees that $\mathcal{M}$ possesses the $\mathbb{Z}_2$ symmetry of the BTI phase. These spaces $\mathcal{M}$ are completely specified by a parametric curve $(r(\sigma), z(\sigma))$, where $r(\sigma)$ is the distance of the surface from the z-axis in $\mathbb{R}^3$ at the height $z(\sigma)$, and $\sigma \in [a, b]$ is a parameter used to specify the curve. If we think of $(r(\sigma), z(\sigma))$ as, say, a curve in the x-z plane (replace r with x), then we can imagine constructing the full surface $\mathcal{M}$ by rotating the curve about the z-axis in $\mathbb{R}^3$. We can then choose coordinates on $\mathcal{M}$ to be (σ, ϕ), where ϕ is the usual azimuthal angle in spherical or cylindrical coordinates in $\mathbb{R}^3$. Finally, in order for this construction to produce a smooth manifold (with no conical singularities at the top and bottom), we require that $\frac{dz}{dr} = 0$ at the top and bottom of the curve. This is equivalent to the condition

$$\frac{\partial_\sigma z(\sigma)|_{\sigma=a,b}}{\partial_\sigma r(\sigma)|_{\sigma=a,b}} = 0\ , \tag{5.3.34}$$

or just

$$\partial_\sigma z(\sigma)|_{\sigma=a,b} = 0\ , \tag{5.3.35}$$

assuming that $\partial_\sigma r(\sigma)$ does not vanish at $\sigma = a, b$.

In principle we can use any parametrization of the surface, but the most convenient choice is a parametrization

[8]We use standard Cartesian coordinates x, y, z for $\mathbb{R}^3$.

$(r(s), z(s))$ in terms of the arc length s along the curve, where

$$s(\sigma) = \int_a^\sigma d\sigma' \sqrt{(\partial_{\sigma'} r(\sigma'))^2 + (\partial_{\sigma'} z(\sigma'))^2} \ . \tag{5.3.36}$$

We define $L = s(b)$ to be the total length of the curve. In the coordinate system (s, ϕ), the metric on $\mathcal{M}$ takes the form

$$g = ds \otimes ds + [r(s)]^2 d\phi \otimes d\phi \ , \tag{5.3.37}$$

and the volume form is

$$\omega_{\mathcal{M}} = r(s) ds \wedge d\phi \ . \tag{5.3.38}$$

The total area of the target space is then $\mathcal{A}_{\mathcal{M}} = 2\pi \int_0^L ds\ r(s)$. In addition, the "unit speed" property $(\partial_s r(s))^2 + (\partial_s z(s))^2 = 1$ of the arc length parametrization, combined with the restriction $\partial_s z(s)|_{s=0,L} = 0$, implies that $\partial_s r(s)|_{s=0} = 1$ and $\partial_s r(s)|_{s=L} = -1$. The signs here follow from the fact that the width of the surface $\mathcal{M}$ *increases* from zero near $s = 0$ and *decreases* back to zero at $s = L$. We also assume that $z(s) = -z(L - s)$ so that $\mathcal{M}$ is symmetric under reflection through the $z = 0$ plane in $\mathbb{R}^3$.

We can now construct a model for the boundary of the BTI using the NLSM with target space $\mathcal{M}$. We denote the NLSM field by $\mathbf{m} = (m^1, m^2)$, with components $m^1 = s$ and $m^2 = \phi$. In the low energy (topological) limit the boundary action contains only a WZ term for $\mathbf{m}$. As usual, to construct this term we require an extension $\mathcal{B}$ of the boundary spacetime S^1_T, and an extension $\tilde{\mathbf{m}}$ of the NLSM field $\mathbf{m}$ into the bulk of $\mathcal{B}$. Then the WZ action describing the low energy physics of the boundary is

$$S_{WZ}[\mathbf{m}] = \frac{2\pi k}{\mathcal{A}_{\mathcal{M}}} \int_{\mathcal{B}} \tilde{\mathbf{m}}^* \omega_{\mathcal{M}} \ , \tag{5.3.39}$$

where $k \in \mathbb{Z}$ is the level of the WZ term. We choose the $U(1)$ and $\mathbb{Z}_2$ symmetries of the BTI state to act on the components of the field $\mathbf{m}$ as

$$U(1) : \phi \to \phi + \xi \ , \tag{5.3.40}$$

and

$$\mathbb{Z}_2 : \phi \quad \to \quad -\phi \tag{5.3.41a}$$

$$s \quad \to \quad L - s \ . \tag{5.3.41b}$$

This action of the $\mathbb{Z}_2$ symmetry is the generalization to the target space $\mathcal{M}$ of the $\mathbb{Z}_2$ action on S^2 from Eqs. (5.3.2).

The next step is to gauge the $U(1)$ symmetry by coupling the boundary WZ action to the gauge field $A = A_t dt$.

One can check that the action

$$S_{WZ,gauged}[\mathbf{m}, A] = S_{WZ}[\mathbf{m}] - \frac{2\pi k}{\mathcal{A}_{\mathcal{M}}} \int_0^T dt\ f(s(t)) A_t \tag{5.3.42}$$

will be invariant under the gauge transformation $\phi \to \phi + \xi$, $A \to A + d\xi$, if the function $f(s)$ satisfies the first order differential equation

$$\partial_s f(s) = r(s)\ . \tag{5.3.43}$$

This equation has the simple solution $f(s) = C + \int_0^s ds'\ r(s')$, where C is an as yet undetermined constant. However, since we require the gauged action to be invariant under the charge-conjugation operation

$$\mathbb{Z}_2 : \phi \quad \to \quad -\phi \tag{5.3.44a}$$

$$s \quad \to \quad L - s \tag{5.3.44b}$$

$$A \quad \to \quad -A\ , \tag{5.3.44c}$$

we find that this constant is fixed to take the value $C = -\frac{\mathcal{A}_{\mathcal{M}}}{4\pi}$. Therefore the function $f(s)$ appearing in the gauged boundary action is given by

$$f(s) = \int_0^s ds'\ r(s') - \frac{\mathcal{A}_{\mathcal{M}}}{4\pi}\ . \tag{5.3.45}$$

In particular we have

$$f(L) = -f(0) = \frac{\mathcal{A}_{\mathcal{M}}}{4\pi}\ , \tag{5.3.46}$$

which will be needed for the calculation of the partition function in the next subsection.

5.3.4 Boundary partition function and global anomaly for all target spaces

We now turn to the evaluation of the partition function $Z[A]$ for the NLSM with target space $\mathcal{M}$ and action given by Eq. (5.3.42) using the equivariant localization (EL) technique. We give a brief introduction to the EL technique in Appendix D.2, and in Appendix D.3 we show how to calculate the Pfaffians which appear in the final expression for $Z[A]$. Therefore, in this section we only outline the calculation and present the result. The final result for the partition function turns out to be *completely identical* to the partition function of Eq. (5.3.28) which we derived for the special case of the $O(3)$ NLSM with target space S^2. The mechanism which underlies the EL technique allows us to understand why this is the case. First, the EL technique applied to our particular problem yields the result that the partition function depends only on field configurations $\mathbf{m}$ near the points on $\mathcal{M}$ which are fixed by the $U(1)$ action. These are just the two points $s = 0$ and $s = L$ at the bottom and the top of $\mathcal{M}$. The value of the gauged WZ action at these two points is actually *independent* of the specific choice of the target space $\mathcal{M}$ (see Eqs. (5.3.52) below). Therefore we find that since the partition function only receives contributions from field configurations near $s = 0$ and $s = L$, and since the action at those two points is independent of the details of $\mathcal{M}$, the partition function $Z[A]$ is independent of the specific details of the target space $\mathcal{M}$. The discussion here is meant to be heuristic, and so we now move on to a more detailed presentation of the calculation.

We start by rewriting the gauged WZ action for the NLSM in a way which makes the problem of computing the partition function of this theory look like a phase space path integral for a dynamical system with phase space $\mathcal{M}$. The reason for this is that the EL technique, in its original formulation, applies to phase space path integrals. To achieve this goal we first recall that we can use a small $U(1)$ gauge transformation to replace the gauge field A_t with its time average $\overline{A}_t$ in the gauged WZ action. Next, we rewrite the gauged WZ action as

$$S_{WZ,gauged}[\mathbf{m}, A] = \int_{\mathcal{B}} \tilde{\mathbf{m}}^* \omega - \int_0^T dt\; H(\mathbf{m}) \;, \tag{5.3.47}$$

where we defined

$$\omega = \frac{2\pi k}{\mathcal{A}_{\mathcal{M}}} \omega_{\mathcal{M}} \tag{5.3.48a}$$

$$H(\mathbf{m}) = \frac{2\pi k}{\mathcal{A}_{\mathcal{M}}} f(s)\overline{A}_t \;. \tag{5.3.48b}$$

We can now see that the path integral for $Z[A]$ is equivalent to a phase space path integral (see our Appendix D.1 for a review) for a dynamical system described by the triple $(\mathcal{M}, \omega, H)$, with the symplectic form ω and Hamiltonian H defined by Eqs. (5.3.48). The Hamiltonian H and the symplectic form ω are related via the equation $dH = -i_{\underline{v}}\omega$, where the vector field $\underline{v}$ is given by

$$\underline{v} = \overline{A}_t \partial_\phi \;. \tag{5.3.49}$$

This vector field is clearly proportional to the vector field ∂_ϕ which generates the action of the $U(1)$ part of the symmetry group $G = U(1) \rtimes \mathbb{Z}_2$ of the BTI on the target space $\mathcal{M}$ of the NLSM. The classical equations of motion for this system are

$$\dot{s} = 0 \tag{5.3.50a}$$

$$\dot{\phi} = \overline{A}_t \;. \tag{5.3.50b}$$

These equations say that (classically) each point on $\mathcal{M}$ revolves around the z-axis in $\mathbb{R}^3$ with a period $\frac{2\pi}{\overline{A}_t}$. In the notation of Appendix D.2 the classical equations of motion can be rewritten as $V^a_S[\mathbf{m}(t); t] = 0$, $a = 1, 2$, where

$V_S^a[\mathbf{m}(t);t] = \dot{m}^a(t) - v^a(\mathbf{m}(t))$ and v^a are the components of the vector field $\underline{v}$ from Eq. (5.3.49).

We are now almost ready to to apply the EL results from Appendix D.2 to compute the partition function. First, let us assume that $T \neq \frac{2\pi n}{\overline{A}_t}$ for any $n \in \mathbb{Z}$. This means that the only T-periodic solutions to the classical equations of motion for the dynamical system defined by $(\mathcal{M}, \omega, H)$ are the *constant* solutions $s = 0$ and $s = L$. Therefore, the set $L\mathcal{M}_S$ of T-periodic solutions to the classical equations of motion (defined in Eq. (D.2.18)) has only these two elements, and the final result for the partition function $Z[A]$ only involves contributions from field configurations close to these solutions. Using the EL technique we find that the partition function can be expressed only in terms of contributions from $s = 0$ and $s = L$ as

$$Z[A] \sim \frac{e^{iS_{WZ,gauged}[\mathbf{m},A]_{s=0}}}{\text{Pf}[\mathcal{O}]_{s=0}} + \frac{e^{iS_{WZ,gauged}[\mathbf{m},A]_{s=L}}}{\text{Pf}[\mathcal{O}]_{s=L}} \ , \tag{5.3.51}$$

where the operator $\mathcal{O}$ is defined in Eq. (D.2.21) of Appendix D.2. The value of the gauged WZ action at these two solutions is

$$S_{WZ,gauged}[\mathbf{m}, A]_{s=0} = \frac{k}{2}\overline{A}_t T \tag{5.3.52a}$$

$$S_{WZ,gauged}[\mathbf{m}, A]_{s=L} = -\frac{k}{2}\overline{A}_t T \ . \tag{5.3.52b}$$

Remarkably, these expressions do not depend on the area $\mathcal{A}_{\mathcal{M}}$, or any other details, of the target space $\mathcal{M}$. We now turn to the evaluation of the Pfaffians appearing in the denominators in Eq. (5.3.51).

To calculate the Pfaffians (which by Eq. (D.2.21) depend on the derivatives of the vector field $\underline{v}$), we first need to express $\underline{v}$ in a system of local coordinates (x, y) near the points $s = 0$ and $s = L$ of the space $\mathcal{M}$. The coordinate system (s, ϕ) is singular at these two points (ϕ is undefined there) and so it cannot be used for an analysis of the space near these two points. Near $s = 0$ we choose coordinates $x = \frac{2\pi k}{\mathcal{A}_{\mathcal{M}}} s\cos(\phi)$, $y = \frac{2\pi k}{\mathcal{A}_{\mathcal{M}}} s\sin(\phi)$, and near $s = L$ we choose coordinates $x = -\frac{2\pi k}{\mathcal{A}_{\mathcal{M}}}(L - s)\cos(\phi)$, $y = \frac{2\pi k}{\mathcal{A}_{\mathcal{M}}}(L - s)\sin(\phi)$. This choice of coordinates has the virtue that the symplectic form ω takes the Darboux form $\omega = dx \wedge dy$ at both $s = 0$ and $s = L$. To derive this result we had to use the important property that $\partial_s r(s)|_{s=0} = -\partial_s r(s)|_{s=L} = 1$. For these choices of coordinates the vector field $\underline{v}$ takes the form

$$\underline{v} = \overline{A}_t(x\partial_y - y\partial_x) \tag{5.3.53}$$

near $s = 0$, and the form

$$\underline{v} = -\overline{A}_t(x\partial_y - y\partial_x) \tag{5.3.54}$$

near $s = L$.

Using the definition of Pf$[\mathcal{O}]$ in terms of a fermion path integral from Eq. (D.2.23) of Appendix D.2, we find that

$\mathrm{Pf}[\mathcal{O}]$ at the points $s = 0$ and $s = L$ is given formally by the determinant of a one-dimensional Dirac operator. More precisely, after expanding the path integral in Fourier modes we find that

$$\begin{aligned}
\mathrm{Pf}[\mathcal{O}]_{s=0} &= -\overline{A}_t \prod_{m>0} \left(\frac{2\pi m}{T} + \overline{A}_t\right)\left(-\frac{2\pi m}{T} + \overline{A}_t\right) \\
&= -\det[-i\partial_t + \overline{A}_t] \ ,
\end{aligned} \tag{5.3.55}$$

and

$$\mathrm{Pf}[\mathcal{O}]_{s=L} = -\det[-i\partial_t - \overline{A}_t] \ . \tag{5.3.56}$$

The operators

$$\mathcal{D}_\pm := -i\partial_t \pm \overline{A}_t \tag{5.3.57}$$

are equivalent to one-dimensional Dirac operators for a fermion in one spacetime dimension coupled to the external field $A = A_t dt$ [76]. As we discussed in Appendix D.2, the overall sign of these Pfaffians is ambiguous, since we are free to alter the order of factors in the definition of the path integral measure. Therefore at this point we are free to choose a particular definition of the path integral measure such that

$$\mathrm{Pf}[\mathcal{O}]_{s=0} = \det[\mathcal{D}_+] \tag{5.3.58}$$

$$\mathrm{Pf}[\mathcal{O}]_{s=L} = \det[\mathcal{D}_-] \ . \tag{5.3.59}$$

These determinants still require proper regularization, and we now turn to a discussion of this issue.

We choose to regularize these determinants using *zeta* and *eta* function methods (see Appendix D.3 for details). To motivate the definition of the regularized determinants in terms of zeta and eta functions, we first consider the following (non-rigorous) manipulations of a definition of these determinants in terms of an infinite product of their eigenvalues. We are also careful to point out any ambiguities which arise in defining the determinants in this way. Let $\lambda_m^{(\pm)} = \frac{2\pi m}{T} \pm \overline{A}_t$, $m \in \mathbb{Z}$, be the eigenvalues of the operator $\mathcal{D}_\pm$. Formally, we have

$$\begin{aligned}
\det[\mathcal{D}_\pm] &= \prod_{m\in\mathbb{Z}} \lambda_m^{(\pm)} \\
&= \prod_{m\in\mathbb{Z}} |\lambda_m^{(\pm)}|\mathrm{sgn}(\lambda_m^{(\pm)}) \ .
\end{aligned} \tag{5.3.60}$$

So far we encounter no difficulties. However, the next step is to express the sign of the eigenvalues as

$$\mathrm{sgn}(\lambda_m^{(\pm)}) = e^{i\frac{\pi}{2}(1-\mathrm{sgn}(\lambda_m^{(\pm)}))} \ . \tag{5.3.61}$$

But this step is ambiguous because we could just as well have written

$$\text{sgn}(\lambda_m^{(\pm)}) = e^{i\frac{(2p+1)\pi}{2}(1-\text{sgn}(\lambda_m^{(\pm)}))} \tag{5.3.62}$$

for *any* integer p. For now we work with the most general expression for $\text{sgn}(\lambda_m^{(\pm)})$, which involves an arbitrary integer p. Later in this section we show how the value of p can be fixed by a minimal number of physical assumptions on the properties of the partition function $Z[A]$.

Continuing with our manipulations, we find that the determinant can be expressed formally as

$$\det[\mathcal{D}_\pm] = \left(\prod_{m\in\mathbb{Z}} |\lambda_m^{(\pm)}|\right) e^{i\frac{(2p+1)\pi}{2}\sum_{m\in\mathbb{Z}}(1-\text{sgn}(\lambda_m^{(\pm)}))} \ . \tag{5.3.63}$$

We now use zeta and eta function methods to make sense of the different terms in this expression. Before we start, we again decompose $\overline{A}_t$ as $\overline{A}_t = \frac{2\pi\ell}{T} + \overline{a}_t$, for some $\ell \in \mathbb{Z}$ and $\overline{a}_t \in (0, \frac{2\pi}{T})$. To start with the regularization, we first use zeta function regularization to define the product over the magnitude of all the eigenvalues $\lambda_m^{(\pm)}$. We carry out this calculation in Appendix D.3 and we find that

$$\left(\prod_{m\in\mathbb{Z}} |\lambda_m|\right)_{reg} = 2\sin\left(\frac{\overline{a}_t T}{2}\right) \ . \tag{5.3.64}$$

Next, we define the sum $\sum_{m\in\mathbb{Z}} 1$ as

$$\left(\sum_{m\in\mathbb{Z}} 1\right)_{reg} = 1 + 2\zeta(0) = 0 \ , \tag{5.3.65}$$

where $\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$ is the Riemann zeta function and we used $\zeta(0) = -\frac{1}{2}$. Finally, we define

$$\left(\sum_{m\in\mathbb{Z}} \text{sgn}(\lambda_m^{(\pm)})\right)_{reg} = \eta_\pm(0) \ , \tag{5.3.66}$$

where $\eta_\pm(0)$ is the analytic continuation to $s = 0$ of the eta function $\eta_\pm(s)$ of the operator $\mathcal{D}_\pm$ (see Appendix D.3 for details). We calculate $\eta_\pm(0)$ in Appendix D.3 and we find that

$$\eta_\pm(0) = \pm 1 \mp \frac{\overline{a}_t T}{\pi} \ . \tag{5.3.67}$$

Putting this all together, we find that the regularized determinants of $\mathcal{D}_{\pm}$ are given by

$$\begin{aligned}\det[\mathcal{D}_{\pm}]_{reg} &= 2\sin\left(\frac{\overline{a}_t T}{2}\right) e^{-i\frac{(2p+1)\pi}{2}(\pm 1\mp\frac{\overline{a}_t T}{\pi})} \\ &= 2(\mp i)^{2p+1}\sin\left(\frac{\overline{a}_t T}{2}\right) e^{\pm i\frac{(2p+1)}{2}\overline{a}_t T} \ ,\end{aligned} \tag{5.3.68}$$

where p was the arbitrary integer which appeared when we tried to rewrite sgn$(\lambda_m^{(\pm)})$ as an exponential. We then find that the partition function for our quantum mechanical system coupled to the external field $A = A_t dt$ evaluates to

$$Z[A] = (-1)^{k\ell+p+1}\frac{\sin\left[\frac{\overline{a}_t T}{2}(k-2p-1)\right]}{\sin\left(\frac{\overline{a}_t T}{2}\right)} \ . \tag{5.3.69}$$

The next step is to determine which choice of p gives the correct partition function. To do this, we will impose the following two conditions on the value of $Z[A=0]$ (the partition function in zero external field). Physically, the value of $Z[A=0]$ is the dimension of the Hilbert space of our quantum mechanical system. Therefore it makes sense to impose the following two conditions on $Z[A=0]$.

1. For $k=0$, we require $Z[A=0]=1$, since $k=0$ gives a trivial theory with action equal to zero. The dimension of the Hilbert space of this theory should be equal to one.

2. For $k\neq 0$, $Z[A=0]$ should be a positive number.

In terms of ℓ and $\overline{a}_t$, the limit $\overline{A}_t\to 0$ is taken by first setting $\ell=0$, and then taking $\overline{a}_t\to 0$. In this limit we find

$$Z[A=0] = (-1)^{p+1}(k-2p-1) \ . \tag{5.3.70}$$

The first condition implies that p satisfies the equation

$$1 = (-1)^p(2p+1) \ . \tag{5.3.71}$$

This equation has the two solutions $p=0$ and $p=-1$. For these two solutions for p, we find that $Z[A=0]$ at any k takes the form

$$Z[A=0] = \begin{cases} -k+1 & , \ p=0 \\ k+1 & , \ p=-1 \end{cases} \ . \tag{5.3.72}$$

We see that in order to satisfy condition two, we must pick $p=-1$ for $k>0$ and $p=0$ for $k<0$. In this way we find that for all k, the partition function is given by

$$Z[A] = (-1)^{k\ell} \frac{\sin\left[\frac{\bar{a}_t T}{2}(|k|+1)\right]}{\sin\left(\frac{\bar{a}_t T}{2}\right)} , \tag{5.3.73}$$

which is identical to the answer we computed for the $O(3)$ NLSM. Therefore we find that for *any* two-dimensional target space $\mathcal{M}$ which respects the symmetries of the BTI phase, the NLSM description of the BTI using the target space $\mathcal{M}$ has *the same* global anomaly as the $O(3)$ NLSM description. This result also implies that a large class of bosonic theories in $0+1$ dimensions with $U(1) \rtimes \mathbb{Z}_2$ symmetry share the same global anomaly as a Dirac fermion in $0+1$ dimensions with the same symmetry [76].

5.4 Renormalization group flows and the fate of our models at low energies

In this section we briefly comment on the expected low energy behavior of the boundary theories discussed in this Chapter. Recall that the basic models we consider are NLSMs with a WZ term. On a d-dimensional spacetime X_{bdy} (which we imagine to lie at the boundary of an SPT phase), we can construct a WZ term for a NLSM with target space $\mathcal{M}$ if dim$[\mathcal{M}] = d+1$. In addition to the WZ term, the NLSM action will also contain an ordinary kinetic term

$$S_{kin}[\mathbf{m}] = \frac{1}{2f} \int_{X_{bdy}} d^d x \, G_{ab}(\mathbf{m}) \partial_\mu m^a \partial^\mu m^b , \tag{5.4.1}$$

where $\mathbf{m} : X_{bdy} \to \mathcal{M}$ is the NLSM field, and $G_{ab}(\mathbf{m})$ is the Riemannian metric on $\mathcal{M}$ (compare with Eq. (5.2.5) for the case of a spherical target space). If we assume that the NLSM field $\mathbf{m}$ is dimensionless, then the coupling constant f has dimensions of $(\text{mass})^{2-d}$. Equivalently, the inverse $\frac{1}{f}$ of the coupling constant has dimensions of $(\text{mass})^{d-2}$. We now consider the consequences of this fact for the low energy behavior of the theories discussed in this Chapter. We focus on the case where $d \geq 2$ since for $d = 1$ our theory is not a quantum field theory but just an ordinary quantum mechanical system.

For simplicity, we first consider the case where the target space $\mathcal{M}$ is the sphere S^{d+1} and so the NLSM field is a $(d+2)$-component unit vector $\mathbf{n}$. In the absence of the WZ term (i.e., for a WZ term with level $k = 0$) then for $d = 2$ the renormalization group (RG) flow is towards the disordered ($f \to \infty$) phase at all scales [42–45]. In this limit the theory is massive and the ground state (or vacuum state) possesses the full $O(d+2) = O(4)$ symmetry of the action (the ground state transforms as a singlet under the action of the $O(4)$ symmetry). When the WZ term is turned on, a stable fixed point appears at a finite value of the coupling f [216], and this fixed point is actually the $SU(2)_k$ Wess-Zumino-Witten conformal field theory. To see this we note that the four-component unit vector field of the $O(4)$ NLSM is equivalent to a 2×2 $SU(2)$ matrix field. Explicitly, if $\mathbf{n} = (n^1, \dots, n^4)$, then one possible mapping to the matrix field U is $U = n^4\mathbb{I} + i\sum_{a=1}^{3} n^a\sigma^a$, where σ^a for $a = 1, 2, 3$, are the Pauli matrices. In addition, the $U(1)$ symmetry that we are interested in in this Chapter is realized as a right (or left, depending on the mapping from $\mathbf{n}$ to U) $U(1)$ symmetry of the $SU(2)_k$ theory, and this symmetry is well-known to be anomalous [39, 40].

For the case of $d > 2$ the coupling constant f is dimensionful and one expects (by a simple power-counting argument) that the theory flows towards the ordered phase $f \to 0$ and so the $O(d+2)$ symmetry of the theory is spontaneously broken at low energies. In fact, for the theory without a topological term, a double perturbation expansion in f and $\epsilon = d - 2$ reveals the existence of an unstable fixed point at a finite value f_1 of the (suitably rescaled) coupling f [43, 44]. If this computation can be trusted, then below this fixed point the theory flows to $f \to 0$ and the symmetry is spontaneously broken, while above the fixed point the theory flows to a (presumably) disordered strong-coupling ($f \to \infty$) phase in which the $O(d+2)$ symmetry is restored[9]. Since in the $d = 2$ case turning on the WZ term introduces a stable fixed point at a finite value of the coupling, some authors have recently proposed a scenario for $d > 2$ in which the introduction of the WZ term introduces a stable fixed point at a finite value $f = f_2$ of the coupling constant, with $f_2 > f_1$, where f_1 is the location of the unstable fixed point (see Figure 2a of Ref. [217]). This possibility was first raised in Ref. [217], and it has been pursued recently in Ref. [218] using a combination of several perturbation expansions. Both of these works consider the case of $d = 3$ spacetime dimensions[10].

What can we deduce about our boundary theories from this discussion? Let us first consider the case for BIQH states. Recall that these boundary theories were $O(2m)$ NLSMs with WZ term in spacetime dimension $2m - 2$, and also NLSMs with deformed target spaces $\mathcal{M}$ which still possessed a $U(1)$ symmetry. We first discuss the case $m > 2$, so that the boundary spacetime dimension is larger than two. In this case, the conclusion which is supported by the most evidence is that the $U(1)$ symmetry of these theories is spontaneously broken in the ground state. In this case the symmetry-broken theory will possess a gapless Goldstone mode. Interestingly, this gapless mode will still couple to the external field $A = A_\mu dx^\mu$ and it is this Goldstone mode which exhibits the anomaly in the symmetry-broken theory. For example, if we consider a general target space $\mathcal{M}$ with $U(1)$ symmetry, and we add a potential to the action which is minimized along the $U(1)$ orbit of a particular point on $\mathcal{M}$, then the low energy theory will possess a gapless Goldstone mode corresponding to motion around this orbit[11].

It is helpful to see an explicit example of this kind in order to appreciate the fact that the Goldstone mode really does exhibit the anomaly. Let us take the $O(2m)$ NLSM with WZ term at level k and introduce a potential into the action which is minimized when $|b_1|^2 = 1$ and all other $b_\ell = 0$ ($\ell = 2, \dots, m$). In the symmetry-broken vacuum we then have $b_1 = e^{i\varphi_{vac}}$ for some constant φ_{vac}. If we expand around this vacuum by setting $b_1 = e^{i\varphi_{vac} + i\varphi}$ then the

[9]The present authors provided further evidence for the existence of this strong coupling phase in our recent work [48], where we computed the beta function for the coupling constant f to leading order in a strong-coupling lattice regularization inspired by the approach of Ref. [45].

[10]More precisely, Ref. [217] considered the $O(4)$ NLSM in $d = 3$ spacetime dimensions with a topological *theta term* with coefficient π. This theory can be understood as a deformation of the $O(5)$ NLSM with WZ term (in the same dimension) at level $k = 1$ in which the fifth component of the NLSM field has been set to zero.

[11]We would like to thank one of the referees of this Chapter for suggesting that we consider an example of this kind.

gauged NLSM action with WZ term reduces to an action for the gapless Goldstone mode φ coupled to A. This action takes the explicit form

$$S[\varphi] = \frac{1}{2f}\int d^{2m-2}x\,(\partial_\mu\varphi - A_\mu)(\partial^\mu\varphi - A^\mu) - \frac{k}{(2\pi)^{m-1}}\int_{X_{bdy}} d\varphi\wedge A\wedge F^{m-2}\,. \tag{5.4.2}$$

It is now easy to see that under a $U(1)$ gauge transformation $\varphi \to \varphi + \chi$, $A \to A + d\chi$, the action for the Goldstone mode φ has the same anomaly as the original $O(2m)$ NLSM. From this analysis we can conclude that even when the $U(1)$ symmetry of the BIQH state is spontaneously broken in the boundary theory, the boundary theory will still possess the same perturbative $U(1)$ anomaly as the original NLSM that we started with.

In the case of $m = 2$ (boundary spacetime dimension equal to two) the situation is more interesting. As we noted above, if we preserve the full $O(4)$ symmetry of the theory, then our theory flows at low energies to the $SU(2)_k$ Wess-Zumino-Witten conformal field theory. On the other hand, we can introduce some $O(4)$-breaking but $U(1)$-preserving anisotropy into the theory to set $|b_1|^2 = 1$ and $b_2 = 0$ (or vice-versa). In this case we end up with a free boson theory of the form

$$S[\varphi] = \frac{1}{2f}\int d^{2}x\,(\partial_\mu\varphi - A_\mu)(\partial^\mu\varphi - A^\mu) - \frac{k}{2\pi}\int_{X_{bdy}} d\varphi\wedge A\,, \tag{5.4.3}$$

where we have $b_1 = e^{i\varphi}$. In this case, however, φ should not be interpreted as a Goldstone boson as we do not have spontaneous symmetry breaking in this dimension. The $SU(2)_k$ theory has a central charge of $c = \frac{3k}{k+2} \geq 1$ (see, for example, Ref. [50]) so it can and will flow to the free boson theory with central charge $c = 1$ when perturbations which break the $O(4)$ symmetry down to $U(1)$ are introduced (this flow is consistent with Zamolodchikov's c-theorem [199]). Note that if we preserve the $U(1)$ symmetry, then the boundary theory cannot be gapped out since we always need some gapless degrees of freedom to saturate the anomaly. Finally, we remark that in the $k = 1$ case, the free boson theory is actually equivalant to the $SU(2)_1$ theory for a particular value of the coupling f. However, marginal perturbations which break the $O(4)$ symmetry down to $U(1)$ will in general tune f away from this special value.

We close this section with a few words about the boundary theories of BTI states. These boundary theories occur in odd spacetime dimensions $2m-1$, and they lie at the boundary of a BTI state in $2m$ dimensions. We have already analyzed the case $m = 1$ in detail in Sec. 5.3. In this case the boundary is just a quantum mechanical system and there are no subtleties involved in assessing the fate of the system at low energies. For the case of $m > 1$, the most likely scenario is that these boundary theories spontaneously break the $U(1) \rtimes \mathbb{Z}_2$ symmetry of the BTI state. As we noted in Ref. [57], because of the way the $U(1)$ symmetry in our models acts on S^{2m} (the target space of the NLSM in this case), in the BTI case it is possible to break the $\mathbb{Z}_2$ symmetry while preserving the $U(1)$ symmetry. In this way we were able to show that the boundary of the BTI state can exhibit a $\mathbb{Z}_2$ symmetry-breaking electromagnetic response, and we found that this response is given by a CS term for A_μ with level $\frac{m!}{2}$. We then argued, based on this evidence, that the boundary theories of the BTI state exhibit a bosonic analogue of the parity anomaly.

For $m > 1$ it is still an open problem to exhibit this bosonic analogue of the parity anomaly in a concrete way (e.g., at the level of the partition function). The most interesting case is $m = 2$ in which the boundary spacetime dimension is $d = 3$. Here we can list three possibilities for the fate of the boundary theory at low energies. First, as noted above, the boundary could break part or all of the symmetry group $U(1) \rtimes \mathbb{Z}_2$ of the BTI state. Second, the results of Refs. [217, 218] indicate that a gapless conformal field theory preserving the full $U(1) \rtimes \mathbb{Z}_2$ symmetry may be possible. Finally, since the anomaly in this case is global and not perturbative, there is the possibility that the boundary theory can flow to a topological quantum field theory whose partition function (in the presence of the external field A_μ) exhibits the anomaly. In this last case all other degrees of freedom at the boundary become gapped and decouple from the topological quantum field theory which describes only the ground state sector of the boundary theory. We comment more on this last possibility in Sec. 5.5.

5.5 Discussion and Conclusion

In this Chapter we continued the program, initiated in Ref. [57], of characterizing the anomalies at the boundary of BIQH and BTI states in all odd and even dimensions, respectively. In Sec. 5.2 we revisited the perturbative $U(1)$ anomaly at the boundary of BIQH states. There we proved that the target space $\mathcal{M}$ of the NLSM describing the boundary theory of these states can be subjected to arbitrary smooth, symmetry-preserving deformations without affecting the anomaly. In Sec. 5.3 we revisited the global anomaly at the boundary of BTI states. In Ref. [57] we gave an argument that the boundary of the BTI state exhibits a bosonic analogue of the parity anomaly of Dirac fermions in odd dimensions. In this Chapter we elevated this argument to a proof for the case of the $(0+1)$-dimensional boundary of the $(1+1)$-dimensional BTI state. In that case we also used the equivariant localization technique to prove that the global anomaly of the BTI boundary is robust against arbitrary smooth, symmetry-preserving deformations of the target space of the NLSM used to describe this state.

From a fundamental point of view, perhaps the most important result in this Chapter is our concrete demonstration, at the level of the partition function, of an analogue of the parity anomaly in a purely bosonic system. Indeed, our result in Sec. 5.3 is a direct bosonic analogue of the results of Ref. [76] on global anomalies of fermions in $0+1$ dimensions. In the context of SPT phases, our results in this Chapter also imply that the universal properties of an SPT phase can be captured by a much wider range of models than the NLSMs with spherical target space originally considered in Refs. [35, 36]. The results of this Chapter lead us to conjecture that an SPT phase in $D+1$ dimensions with symmetry group G, which would be described by an $O(D+2)$ NLSM in the approach of Refs. [35, 36], can be modeled using an

NLSM with *any* target space $\mathcal{M}$ related to S^{D+1} by a diffeomorphism which is equivariant with respect to the action of the group G. Note that this conjecture only applies to SPT phases for which an NLSM description *exists*. This does not seem to be the case for all SPT (or short-range entangled) phases, for example the "E8" state in $2+1$ dimensions and the "beyond cohomology" state with time-reversal symmetry in $3+1$ dimensions [9, 19].

An ambitious goal for future work would be to present a concrete demonstration, again at the level of the partition function, of an analogue of the parity anomaly in a $(2+1)$-dimensional bosonic model with $U(1)$ and $\mathbb{Z}_2$ symmetry, where $\mathbb{Z}_2$ now represents time-reversal. A precise understanding of global anomalies in $(2+1)$-dimensional bosonic systems would also be extremely useful in the search for new dualities in quantum field theory in $2+1$ dimensions [141–144, 146, 147, 219]. A crucial check on any proposed duality is that the two theories which are conjectured to be dual to each other must have the same 't Hooft anomalies when coupled to various external fields.

An interesting candidate for a $(2+1)$-dimensional bosonic model displaying a bosonic analogue of the parity anomaly is the $O(5)$ NLSM with WZ term, and with the $U(1) \rtimes \mathbb{Z}_2$ symmetry of the BTI state acting in the manner described in Ref. [57]. In Ref. [57] we already gave several pieces of evidence which suggest that this model displays a bosonic analogue of the parity anomaly. The first piece of evidence was our computation of the time-reversal breaking electromagnetic response of this model, which we already mentioned above. However, we also gave a second argument which was based on the demonstration that there is a certain composite vortex excitation in this model with fermionic statistics (an observation which goes back to Refs. [19, 200]), and such an excitation should not exist in a purely bosonic model which is not anomalous.

The $O(5)$ NLSM with WZ term may be tractable analytically in the topological limit in which the coupling constant f_{bdy} of the NLSM is sent to infinity (i.e., if one considers the model with only the topological term). This would correspond to the third possibility that we raised at the end of Sec. 5.4: the boundary theory could flow to a topological quantum field theory whose partition function exhibits the anomaly. It may even be the case that a more sophisticated version of the equivariant localization technique can be used to calculate the partition function of the $O(5)$ NLSM with WZ term in the topological limit and properly coupled to an external $U(1)$ gauge field as described in Sec. VI of Ref. [57]. However, there are several difficulties which must be surmounted before one can apply any kind of equivariant localization technique to this problem. The main problem is that one needs to find a hidden supersymmetry in this problem which can be exploited in order to establish the localization of the path integral. In the $(0+1)$-dimensional case this supersymmetry followed, at least partially[12], from the fact that the path integral measure could be exponentiated by introducing a set of *real* Grassmann-valued (i.e., fermionic) fields $\eta^a(t)$ into the problem. This could only be done with real fermionic fields because the target spaces of the $(0+1)$-dimensional NLSMs that we studied were all symplectic manifolds. On the other hand, the target space S^4 of the $O(5)$ NLSM

[12] As we reviewed in Appendix D.2, the fact that the Hamiltonian was associated with a $U(1)$-action on the phase space was also a crucial ingredient.

is not symplectic. Therefore one can only exponentiate the path integral measure by introducing complex fermionic fields. Currently, we are not aware of a generalization of the equivariant localization techniques of Refs. [204–207] which starts by exponentiating the path integral measure by introducing complex fermions, but such a generalization may still be possible. We leave a detailed investigation of this to future work.

Appendix A

Supplement to Chapter 2

A.1 Computation of the Path-Ordered Integral

To calculate the matrix $\mathcal{R}(t)$, which gives the rigid motion of the swimmer after a finite time t, we need to evaluate the reverse path-ordered integral (2.2.6). In practice we do this by slicing time into many small steps (say N steps) of size Δt. We can write

$$\bar{P}e^{\int_0^t \mathcal{A}(t')dt'} = \bar{P}e^{\sum_{i=1}^{N} \int_{(i-1)\Delta t}^{i\Delta t} \mathcal{A}(t')dt'} . \tag{A.1.1}$$

If the time steps are small enough then we can approximate this as

$$\bar{P}e^{\sum_{i=1}^{N} \int_{(i-1)\Delta t}^{i\Delta t} \mathcal{A}(t')dt'} \approx \prod_{i=1}^{N} \bar{P}e^{\int_{(i-1)\Delta t}^{i\Delta t} \mathcal{A}(t')dt'} , \tag{A.1.2}$$

where on the right side we now have a product of reverse path-ordered integrals over many small time intervals of size Δt and we should put the earliest times on the right so that we are applying the rigid motions in these small intervals in chronological order. Since these time intervals are very small we can make a further approximation by expanding the path-ordered integral over the time interval Δt to first order and neglecting the higher order terms to find:

$$\bar{P}e^{\int_{(i-1)\Delta t}^{i\Delta t} \mathcal{A}(t')dt'} \approx I + \int_{(i-1)\Delta t}^{i\Delta t} \mathcal{A}(t')dt' . \tag{A.1.3}$$

Finally we can make one further approximation for the integral of the matrix $\mathcal{A}(t)$ over the small time interval Δt,

$$\int_{t_{i-1}}^{t_i} \mathcal{A}(t')dt' \approx \mathcal{A}(t_{i-1})\Delta t \tag{A.1.4}$$

where $t_i = i\Delta t$ (and $t_0 = 0$). Our final expression for the approximation of the full path-ordered integral is then

$$\bar{P}e^{\int_0^t \mathcal{A}(t')dt'} \approx \prod_{i=1}^{N} \left(I + \mathcal{A}(t_{i-1})\Delta t\right) , \tag{A.1.5}$$

where again the matrices for the earliest times must be to the right so that the rigid motions are applied in the proper order.

We have also tried expanding the reverse path-ordered integrals over the time interval Δt to second order, but it seems that this makes almost no visible correction to the swimming trajectory when the swimming deformations are not too large and the step size Δt is small.

A.2 Proof that Re$[b_{-2}] = \frac{1}{2\pi}\frac{dA(t)}{dt}$ for general swimming strokes

Our analysis of the simple swimmer (2.5.1) suggests a deeper connection between the area of the swimmer and the odd viscosity contribution to the torque on the swimmer. To explore this connection further we now show that Eq. (2.6.11) holds for any swimmer whose boundary is a smooth curve without self-intersections.

The boundary of the swimmer is just a smooth curve parameterized by θ which also depends on the time t. If we write the shape in terms of real components

$$S_0(\theta, t) = x(\theta, t) + iy(\theta, t) \tag{A.2.1}$$

and plug into the area formula (2.5.2) we find

$$A(t) = \frac{1}{2}\int_0^{2\pi} \left[x(\theta, t)y'(\theta, t) - y(\theta, t)x'(\theta, t)\right] d\theta \ , \tag{A.2.2}$$

where the prime denotes a derivative with respect to θ. Next take a time derivative to get

$$\frac{dA(t)}{dt} = \frac{1}{2}\int_0^{2\pi} \Big[\dot{x}(\theta, t)y'(\theta, t) + x(\theta, t)\dot{y}'(\theta, t) - \dot{y}(\theta, t)x'(\theta, t) - y(\theta, t)\dot{x}'(\theta, t)\Big] d\theta \ . \tag{A.2.3}$$

We can integrate by parts on the terms with mixed partial derivatives and use the fact that the boundary terms vanish since $x(\theta, t)$, $y(\theta, t)$, $\dot{x}(\theta, t)$ and $\dot{y}(\theta, t)$ are 2π-periodic in θ to get

$$\frac{dA(t)}{dt} = \int_0^{2\pi} \left[\dot{x}(\theta, t)y'(\theta, t) - \dot{y}(\theta, t)x'(\theta, t)\right] d\theta \ . \tag{A.2.4}$$

Because of the no-slip boundary conditions the vector $(\dot{x}(\theta, t), \dot{y}(\theta, t))$ is just the fluid velocity $\mathbf{v}(\mathbf{r})$ evaluated on the surface of the swimmer,

$$\mathbf{v}(\mathbf{r})|_{\text{swimmer}} = \dot{x}(\theta, t)\hat{\mathbf{x}} + \dot{y}(\theta, t)\hat{\mathbf{y}} \ . \tag{A.2.5}$$

Then we can write

$$\frac{dA(t)}{dt} = \oint_{\text{swimmer}} \mathbf{v} \cdot \hat{\mathbf{n}}\, ds = \Phi(\text{swimmer}) \,, \tag{A.2.6}$$

where $\hat{\mathbf{n}}\, ds = d\mathbf{r} \times \hat{\mathbf{z}}$ is a vector normal to the surface of the swimmer with magnitude $ds = |d\mathbf{r}|$. This integral is just the flux of the fluid at the surface of the swimmer. By the divergence theorem we have

$$\Phi(\infty) - \Phi(\text{swimmer}) = \int_{\text{fluid}} \nabla \cdot \mathbf{v}\, dxdy \tag{A.2.7}$$

and since the fluid is incompressible, $\nabla \cdot \mathbf{v} = 0$, we get

$$\frac{dA(t)}{dt} = \Phi(\infty) \,. \tag{A.2.8}$$

A comparison with Eq. (2.4.15) for the flux of the fluid at infinity yields the final result

$$\text{Re}[b_{-2}] = \frac{1}{2\pi} \frac{dA(t)}{dt} \,, \tag{A.2.9}$$

proving that this relation is valid for general swimming shapes in incompressible fluids.

It is known that an object which rotates in a fluid with odd viscosity will feel a pressure directed radially inwards or outwards depending on the direction of the rotation [23]. The relation (2.6.11) is the companion to this statement. It says that an object which tries to expand or contract in a fluid with odd viscosity will feel a torque whose direction ($\pm\hat{\mathbf{z}}$) depends on whether the area of the object is growing or shrinking.

A.3 Extension and solution of conformal maps of degree $\mathcal{D} = 3$

A swimmer whose boundary is a degree 3 ($\mathcal{D} = 3$) conformal map of the circle has the form

$$S_0(\sigma, t) = \alpha_0(t)\sigma + \alpha_{-2}(t)\sigma^{-1} + \alpha_{-3}(t)\sigma^{-2} + \alpha_{-4}\sigma^{-3} \,, \tag{A.3.1}$$

with area

$$A(t) = \pi(|\alpha_0|^2 - |\alpha_{-2}|^2 - 2|\alpha_{-3}|^2 - 3|\alpha_{-4}|^2) \,. \tag{A.3.2}$$

To solve for a^*_{-1} and b^*_{-2} we need the coefficients a^*_{-2}, a^*_{-3} and a^*_{-4}. Equations for these coefficients can be obtained by plugging into the pulled-back velocity expansion (2.6.3). We find that

$$\alpha_{-4}\bar{a}^*_{-2} + \bar{\alpha}_0 a^*_{-2} = \bar{\alpha}_0\dot{\alpha}_{-2} \tag{A.3.3a}$$

$$a^*_{-3} = \dot{\alpha}_{-3} \tag{A.3.3b}$$

$$a^*_{-4} = \dot{\alpha}_{-4} \; . \tag{A.3.3c}$$

The equation for a^*_{-2} is really just a matrix equation for a two-component vector consisting of the real and imaginary parts of a^*_{-2}. The solution is

$$a^*_{-2} = \frac{|\alpha_0|^2\dot{\alpha}_{-2} - \alpha_0\alpha_{-4}\dot{\bar{\alpha}}_{-2}}{|\alpha_0|^2 - |\alpha_{-4}|^2} \; . \tag{A.3.4}$$

In terms of this coefficient we find that

$$a_{-1} = -(\bar{\alpha}_0)^{-1}(\bar{a}^*_{-2}\alpha_{-3} + 2\dot{\alpha}_{-3}\alpha_{-4}) \tag{A.3.5}$$

and

$$b_{-2} = \bar{\alpha}_0\dot{\alpha}_0 - \bar{\alpha}_{-2}\dot{\alpha}_{-2} - 2\alpha_{-3}\dot{\bar{\alpha}}_{-3} - 3\alpha_{-4}\dot{\bar{\alpha}}_{-4} + \bar{\alpha}_{-2}a^*_{-2} - \alpha_{-2}\bar{a}^*_{-2} \; . \tag{A.3.6}$$

Note that the last two terms in b_{-2} are complex conjugates of each other and appear with the opposite sign so that they will cancel when we take the real part of b_{-2}. This means that the relation $\mathrm{Re}[b_{-2}] = \frac{1}{2\pi}\frac{dA(t)}{dt}$ still holds in this case, as we expect based on the general arguments presented in Appendix A.2.

To solve for the new form of the angular velocity necessary to cancel the torque on the swimmer, we again send $\alpha_i \to \alpha_{i,rot} = \alpha_i e^{i\omega t}$ and solve the equation

$$\eta^e \mathrm{Im}[b_{-2,rot}] - \eta^o \mathrm{Re}[b_{-2,rot}] = 0 \; , \tag{A.3.7}$$

where now

$$b_{-2,rot} = b_{-2} + i\omega J \tag{A.3.8}$$

with

$$J = \frac{2}{|\alpha_0|^2 - |\alpha_{-4}|^2}\left(|\alpha_0|^2|\alpha_{-2}|^2 + \mathrm{Re}[\alpha_0\alpha_{-4}(\bar{\alpha}_{-2})^2]\right) + |\alpha_0|^2 - |\alpha_{-2}|^2 + 2|\alpha_{-3}|^2 + 3|\alpha_{-4}|^2 \; . \tag{A.3.9}$$

The new angular velocity needed to cancel the torque on the swimmer is then

$$\omega = \frac{1}{J}\left(-\mathrm{Im}[b_{-2}] + \tfrac{\eta^o}{\eta^e}\mathrm{Re}[b_{-2}]\right) \; , \tag{A.3.10}$$

so that the translational and rotational parts of the gauge potential are now given by

$$\mathcal{A}_{tr} = (\bar{\alpha}_0)^{-1}(\bar{a}^*_{-2}\alpha_{-3} + 2\dot{\alpha}_{-3}\alpha_{-4}) \tag{A.3.11}$$

$$\mathcal{A}_{rot} = \frac{1}{J}(-\mathrm{Im}[b_{-2}] + \tfrac{\eta^o}{\eta^e}\mathrm{Re}[b_{-2}])\ . \tag{A.3.12}$$

A.4 Uniqueness Theorem for Slow Flow Equations with Odd Viscosity

The slow flow equations for fluids with odd viscosity are

$$\nabla(p - \eta^o\xi) = \eta^e\nabla^2\mathbf{v} \tag{A.4.1}$$

$$\nabla\cdot\mathbf{v} = 0\ , \tag{A.4.2}$$

where $\xi = (\nabla\times\mathbf{v})\cdot\hat{\mathbf{z}}$ is the vorticity. We first prove uniqueness of the solution in the case where the swimmer occupies a region S, bounded by a smooth closed curve ∂S. The fluid occupies the space in the plane outside of the swimmer, which we denote by $\mathbb{R}^2 \setminus S$. The proof here is very similar to the proof one uses in the even viscosity case when no-slip boundary conditions are imposed on a smooth closed curve (for that proof see Ref. [97]).

Suppose we have two solutions, $\mathbf{v}_1$ and $\mathbf{v}_2$, to the above equations that both satisfy the no-slip boundary conditions

$$\mathbf{v}|_{\partial S} = \mathbf{v}_0 \tag{A.4.3}$$

on the surface of the swimmer. Construct the difference of the two velocity fields, $\mathbf{V} = \mathbf{v}_1 - \mathbf{v}_2$, and the differences of the pressures and vorticities $P = p_1 - p_2$, $\Xi = \xi_1 - \xi_2$. Since the slow-flow equations are linear, these quantities satisfy the equations

$$\nabla(P - \eta^o\Xi) = \eta^e\nabla^2\mathbf{V} \tag{A.4.4}$$

$$\nabla\cdot\mathbf{V} = 0\ , \tag{A.4.5}$$

but with the boundary condition

$$\mathbf{V}|_{\partial S} = \mathbf{0}\ . \tag{A.4.6}$$

We want to show that this boundary condition forces $\mathbf{V} = \mathbf{0}$ everywhere inside the fluid. Take the dot product of the first equation with $\mathbf{V}$ and integrate both sides over the region containing the fluid,

$$\iint_{\mathbb{R}^2 \setminus S} \mathbf{V} \cdot \nabla(P - \eta^o \Xi) dA = \eta^e \iint_{\mathbb{R}^2 \setminus S} \mathbf{V} \cdot \nabla^2 \mathbf{V} dA \,. \tag{A.4.7}$$

Since $\nabla \cdot \mathbf{V} = 0$, the integrand on the left-hand side can be written as

$$\mathbf{V} \cdot \nabla(P - \eta^o \Xi) = \nabla \cdot [(P - \eta^o \Xi)\mathbf{V}] \,. \tag{A.4.8}$$

We can then use the divergence theorem on the left side, so that our relation becomes

$$\oint_{\partial S} (P - \eta^o \Xi) \mathbf{V} \cdot \hat{\mathbf{n}} ds = \eta^e \iint_{\mathbb{R}^2 \setminus S} \mathbf{V} \cdot \nabla^2 \mathbf{V} dA \,, \tag{A.4.9}$$

where $\hat{\mathbf{n}}$ is the unit normal vector to the curve ∂S. But $\mathbf{V} = \mathbf{0}$ on ∂S, so the integral on the left-hand side is zero, and we just get

$$\iint_{\mathbb{R}^2 \setminus S} \mathbf{V} \cdot \nabla^2 \mathbf{V} dA = 0 \,. \tag{A.4.10}$$

From now on it will be more useful to write everything out in coordinates (but not using the summation convention). Let $\mathbf{V} = (V_1, V_2)$. The velocity is a function of position $\mathbf{x} = (x_1, x_2)$ inside the fluid. We have

$$\mathbf{V} \cdot \nabla^2 \mathbf{V} = \sum_i V_i \left(\sum_j \frac{\partial^2 V_i}{\partial x_j^2} \right) \,. \tag{A.4.11}$$

We can use the chain rule to rewrite this as

$$\mathbf{V} \cdot \nabla^2 \mathbf{V} = \sum_{i,j} \left[\frac{\partial}{\partial x_j} \left(V_i \frac{\partial V_i}{\partial x_j} \right) \right] - \sum_{i,j} \left(\frac{\partial V_i}{\partial x_j} \right)^2 \,. \tag{A.4.12}$$

If we define the vector $\mathbf{W}$ with components

$$W_j = \sum_i V_i \frac{\partial V_i}{\partial x_j} \tag{A.4.13}$$

then we can write this compactly as

$$\mathbf{V} \cdot \nabla^2 \mathbf{V} = \nabla \cdot \mathbf{W} - \sum_{i,j} \left(\frac{\partial V_i}{\partial x_j} \right)^2 \,. \tag{A.4.14}$$

Now

$$\iint_{\mathbb{R}^2 \setminus S} \nabla \cdot \mathbf{W} dA = \oint_{\partial S} \mathbf{W} \cdot \hat{\mathbf{n}} ds = 0 \,, \tag{A.4.15}$$

where we have again used the fact that $\mathbf{V}$ vanishes on ∂S. So we are left with

$$\iint_{\mathbb{R}^2 \smallsetminus S} \sum_{i,j} \left(\frac{\partial V_i}{\partial x_j} \right)^2 dA = 0 \,. \tag{A.4.16}$$

But the integrand in this expression is greater than or equal to zero, so we conclude that

$$\frac{\partial V_i}{\partial x_j} = 0 \,\forall\, i, j \tag{A.4.17}$$

so $\mathbf{V}$ is a constant independent of position. But $\mathbf{V} = \mathbf{0}$ on the surface of the swimmer, therefore $\mathbf{V} = \mathbf{0}$ everywhere, and so $\mathbf{v}_1 = \mathbf{v}_2$. The solution is unique.

A.4.1 Possible Proof for Swimmers of More General Shapes

Now we consider swimmers of a more general shape. We can imagine two situations here. In the first situation the swimmer is bounded by a closed curve which might not be smooth, for example a swimmer with a "blocky" shape. In the second situation we have a very thin swimmer whose entire body consists of a one-dimensional curve, open on both ends, and not necessarily smooth. An example of this situation is our simple model of the scallop in Fig. 2.5a of Sec. 2.8. In both cases we call the curve Γ, and we impose no-slip boundary conditions for the fluid on this curve,

$$\mathbf{v}|_\Gamma = \mathbf{v}_0 \,. \tag{A.4.18}$$

Define

$$v_{max} = \max \left\{ |\mathbf{v}(\mathbf{x})| : \mathbf{x} \in \Gamma \right\} \,. \tag{A.4.19}$$

It is the speed of the fastest moving point on the swimmer.

Next, we recall the physical meaning of the slow flow equations. The full Navier-Stokes equations (for an incompressible fluid with odd viscosity) are

$$\rho \left(\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla)\mathbf{v} \right) = \eta^e \nabla^2 \mathbf{v} - \nabla(p - \eta^o \xi) \tag{A.4.20}$$

$$\nabla \cdot \mathbf{v} = 0 \,. \tag{A.4.21}$$

The slow flow equations are obtained from these by setting the convective derivative term to zero,

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla)\mathbf{v} = 0 \,. \tag{A.4.22}$$

The physical meaning of this statement is that the net force on each fluid element, represented by the right-hand side of Eq. (A.4.20), is equal to zero. The fact that the convective derivative is equal to zero means that the fluid velocity $\mathbf{v}$ is a constant along streamlines in the fluid. With an object moving in the fluid (with no-slip boundary conditions), the streamlines must begin on that object, and they either end on that object as well (if the flux of the fluid at infinity vanishes), or the streamlines go out to infinity (if the flux of the fluid at infinity does not vanish). Furthermore, every point in the fluid lies on exactly one streamline (there are no shocks in this situation).

These considerations imply that v_{max} is actually an upper bound for the speed of the fluid anywhere in the entire plane $\mathbb{R}^2$. So we can say that

$$|\mathbf{v}(\mathbf{x})| \leq v_{max} \; \forall \, \mathbf{x} \, . \tag{A.4.23}$$

We can now use this fact to prove the uniqueness of solutions to the slow flow equations for more general shapes, including our non-smooth, possibly open-ended, curve Γ. Again, suppose we have two solutions $\mathbf{v}_1$ and $\mathbf{v}_2$, both satisfying the boundary condition $\mathbf{v}|_\Gamma = \mathbf{v}_0$. Then the difference $\mathbf{V} = \mathbf{v}_1 - \mathbf{v}_2$ again satisfies the slow flow equations, but with the boundary condition $\mathbf{V}|_\Gamma = \mathbf{0}$. For this boundary condition we have $v_{max} = 0$. Then our bound Eq. (A.4.23) implies that $|\mathbf{V}| \leq 0$ everywhere, so we can conclude that $\mathbf{V} = \mathbf{0}$ everywhere. This implies that $\mathbf{v}_1 = \mathbf{v}_2$, so the solution is unique.

Since we used the vanishing of the convective derivative in the derivation, the bound Eq. (A.4.23) applies only to time-independent viscous flows, and almost certainly doesn't apply at all to any other kinds of fluid flows.

Appendix B

Supplement to Chapter 3

B.1 Quantum generators of the $U(N)$ action

In this Appendix we consider the form of the quantum generators of the $U(N)$ transformations of the matrix model variables X^a and Ψ. We use this result in Sec. 3.4 to show that the constraint of Eq. (3.4.6) simply forces physical states in the CSMM to be singlets under the $SU(N)$ action, and to carry a certain total charge under the $U(1)$ action. This information is sufficient to write down a basis of physical states (states respecting the constraint) for the model following Ref. [117].

We start with the generators for the $U(N)$ transformation of the complex vector variable Ψ. Under a $U(N)$ transformation by a matrix V we have $\Psi \to V\Psi$ or in components

$$\Psi^j \to V^j{}_k \Psi^k \; . \tag{B.1.1}$$

We are interested in the infinitesimal form of this transformation, so we take $V = e^{iT}$ for a Hermitian matrix T (the Lie algebra of the group $U(N)$ consists of the $N \times N$ Hermitian matrices). Then to first order in T we have $\Psi \to \Psi + iT\Psi$. In components, the first order change in Ψ^j generated by T is

$$\delta_T \Psi^j = iT^j{}_k \Psi^k \; . \tag{B.1.2}$$

We now look for a quantum operator $\mathcal{O}_\Psi(T)$ such that

$$[\mathcal{O}_\Psi(T), \Psi^j] = iT^j{}_k \Psi^k \; , \tag{B.1.3}$$

i.e., the quantum commutator of $\mathcal{O}_\Psi(T)$ with Ψ^j implements the infinitesimal $U(N)$ action generated by T (this is what we mean when we say that a quantum operator *generates* the $U(N)$ action). The correct operator is (in terms of b^j instead of Ψ^j)

$$\mathcal{O}_\Psi(T) = -ib^\dagger_j T^j{}_k b^k \; . \tag{B.1.4}$$

Thus, $\mathcal{O}_\Psi(T)$ is the quantum operator which generates the $U(N)$ transformation $V = e^{iT}$ acting on Ψ. One can also check that the operators $\mathcal{O}_\Psi(T)$ obey the Lie algebra of $U(N)$. To check this it is sufficient to check that the map $T \mapsto \mathcal{O}_\Psi(T)$ is a Lie algebra homomorphism, i.e., that

$$[\mathcal{O}_\Psi(T_1), \mathcal{O}_\Psi(T_2)] = \mathcal{O}_\Psi(-i[T_1, T_2]_M) \ , \tag{B.1.5}$$

and it is straightforward to verify that this relation holds for our generators $\mathcal{O}_\Psi(T)$.

Next we consider the matrix variables X^a. Under a $U(N)$ transformation we have $X^a \to V X^a \overline{V}^T$. Writing $V = e^{iT}$ as before, we find that to first order in T we have $X^a \to X^a + i[T, X^a]_M$. Note that for $T = \alpha\mathbb{I}$, i.e., for $U(1)$ transformations, the matrix variables X^a are invariant. Therefore we can restrict our attention to $SU(N)$ transformations for the X^a variables. We then choose T to be one of the generators T^A of $SU(N)$, and examine the infinitesimal action of $V = e^{iT^A}$ on the scalar variables x_0^a and x_A^a, $A = 1, \ldots, N^2-1$, which appear in the expansion of X^a from Eq. (3.4.11). We have

$$\begin{aligned} \delta_{T^A} X^a &= i[T^A, X^a]_M \\ &= i \sum_{B=1}^{N^2-1} x_B^a [T^A, T^B]_M \\ &= -\sum_{B,C=1}^{N^2-1} x_B^a f^{ABC} T^C \ . \end{aligned} \tag{B.1.6}$$

From this we read off that $\delta_{T^A} x_0^a = 0$ (reflecting the invariance under $U(1)$ transformations), and

$$\delta_{T^A} x_B^a = -\sum_{C=1}^{N^2-1} x_C^a f^{ACB} \ , \ B = 1, \ldots, N^2-1 \ . \tag{B.1.7}$$

We now look for a quantum operator $\mathcal{O}_X(T^A)$ which generates this action on the variables x_A^a ($A = 1, \ldots, N^2 - 1$), i.e., an operator which commutes with x_0^a and satisfies

$$[\mathcal{O}_X(T^A), x_B^a] = -\sum_{C=1}^{N^2-1} x_C^a f^{ACB} \tag{B.1.8}$$

for $B = 1, \ldots, N^2 - 1$. One can check that the correct operator is (in terms of the oscillator variables a_A)

$$\mathcal{O}_X(T^A) = \sum_{B,C=1}^{N^2-1} f^{ACB} a_B^\dagger a_C \ . \tag{B.1.9}$$

This completes the construction of the quantum generators of the $U(N)$ action on the X^a and Ψ variables in the

CSMM. This is all the information which is needed to analyze the $j \neq k$ elements of the CSMM constraint $G^j{}_k$ from Eq. (3.4.29).

B.2 Kubo formula approach to Hall viscosity in the CSMM

In this Appendix we use a Kubo formula approach inspired by Ref. [92] to compute the Hall viscosity in the ground state of the CSMM. For this computation we subject the CSMM to a time-dependent APD (or strain) parametrized by $\alpha_{ab}(t)$ such that the dynamics of the system is described by the time-dependent Hamiltonian

$$H(\alpha(t)) = U(\alpha(t)) H_{CSMM} U(\alpha(t))^\dagger \ . \tag{B.2.1}$$

Here the operator $U(\alpha(t))$ is the APD generator for the CSMM which we derive in Sec. 3.5 of the main text. We also assume that at the time t_0 we have $\alpha_{ab}(t_0) = 0$ so that $|\psi(t_0)\rangle = |\psi_0\rangle$, which is the ground state of the CSMM from Eq. (3.4.39). As we discussed in Sec. 3.2, the generalized force associated with the APD parametrized by the coefficients α_{ab} is

$$F^{ab} = -\frac{\partial H(\alpha)}{\partial \alpha_{ab}}\Big|_{\alpha=0} = -i[\Lambda^{ab}, H_{CSMM}] \ . \tag{B.2.2}$$

To calculate the Hall viscosity we need to compute the expectation value of the generalized force F^{ab} in the state $|\psi(t)\rangle$ of the system, where $|\psi(t)\rangle$ is the solution to the time-dependent Schrodinger equation

$$H(\alpha(t))|\psi(t)\rangle = i\hbar \frac{\partial}{\partial t}|\psi(t)\rangle \ . \tag{B.2.3}$$

We now discuss the details of this computation.

First, to set up this problem in a form which is amenable to perturbation theory and the Kubo formula, we make a time-dependent change of states by writing

$$|\psi(t)\rangle = U(\alpha(t))|\phi(t)\rangle \ . \tag{B.2.4}$$

The state $|\phi(t)\rangle$ is then the solution to a time-dependent Schrodinger equation with a new Hamiltonian $H'(t)$ given by

$$H'(t) = H_{CSMM} + V(t) \tag{B.2.5}$$

with

$$
\begin{aligned}
V(t) &= -i\hbar U(\alpha(t))^\dagger \frac{\partial U(\alpha(t))}{\partial t} \\
&\approx \hbar \frac{\partial \alpha_{ab}(t)}{\partial t}\Lambda^{ab} + \dots \ ,
\end{aligned}
\tag{B.2.6}
$$

where in the second line we expanded the perturbation $V(t)$ to first order in $\alpha_{ab}(t)$. The new Hamiltonian $H'(t)$ is now expressed as a time-independent term H_{CSMM} plus a time-dependent perturbation $V(t)$, and is therefore in a form[1] which is amenable to an application of standard linear response theory.

To compute the Hall viscosity we naively want to compute the expectation value of F^{ab} in the state $|\psi(t)\rangle$. However, in Ref. [92] the authors argued that one should instead compute the expectation value of $U(\alpha(t))F^{ab}U(\alpha(t))^\dagger$, which is equivalent to expressing the generalized force F^{ab} in terms of the strained coordinates $U(\alpha(t))x^a_A U(\alpha(t))^\dagger$ instead of the original coordinates x^a_A of the CSMM (in the language of Ref. [92], we express the generalized force in terms of the "**X**" variables as opposed to the unstrained "**x**" variables). The reason for this is as follows. We view the APD parametrized by $\alpha_{ab}(t)$ as an active transformation (i.e., we physically deform the fluid/CSMM), and so in the computation of the response to this APD we should use the generalized force expressed in terms of the coordinates of the deformed system. Now we have

$$
\langle\psi(t)|U(\alpha(t))F^{ab}U(\alpha(t))^\dagger|\psi(t)\rangle = \langle\phi(t)|F^{ab}|\phi(t)\rangle \ , \tag{B.2.7}
$$

and so it remains to compute the expectation value $\langle\phi(t)|F^{ab}|\phi(t)\rangle$.

In interaction picture perturbation theory in the strength of the potential $V(t)$, the expectation value of any time-independent operator A in the state $|\phi(t)\rangle$ is given by the standard Kubo formula as

$$
\langle\phi(t)|A|\phi(t)\rangle - \langle\phi(t_0)|A|\phi(t_0)\rangle = -\frac{i}{\hbar}\int_{t_0}^{t} dt' \ \langle\phi(t_0)|[A_I(t), V_I(t')]|\phi(t_0)\rangle + \dots \ , \tag{B.2.8}
$$

where $A_I(t) = e^{i\frac{H_{CSMM}(t-t_0)}{\hbar}} A e^{-i\frac{H_{CSMM}(t-t_0)}{\hbar}}$ is in the interaction picture defined by evolution with H_{CSMM}, and likewise for $V_I(t') = e^{i\frac{H_{CSMM}(t'-t_0)}{\hbar}} V(t') e^{-i\frac{H_{CSMM}(t'-t_0)}{\hbar}}$. Note also that for any time-independent A we have $A_I(t_0) = A$, and we also have $|\phi(t_0)\rangle = |\psi(t_0)\rangle = |\psi_0\rangle$.

For the application to the calculation of the Hall viscosity we set $A = F^{ab}$ and keep only the term in $V(t)$ which

[1] The change of basis from $|\psi(t)\rangle$ to $|\phi(t)\rangle$ is equivalent to the change from the "**x**" to the "**X**" variables in Ref. [92]. We thank Barry Bradlyn for helpful discussions on this point.

is linear in the parameters $\alpha_{ab}(t)$. This yields the expression

$$\langle F^{ab}\rangle_t - \langle F^{ab}\rangle_{t_0} = -i\int_{t_0}^{t} dt'\ \langle[F_I^{ab}(t), \Lambda_I^{cd}(t')]\rangle_{t_0}\frac{\partial\alpha_{cd}(t')}{\partial t'}\ , \tag{B.2.9}$$

where we used the shorthand notation $\langle F^{ab}\rangle_t \equiv \langle\phi(t)|F^{ab}|\phi(t)\rangle$, etc. Next, since $\langle[F_I^{ab}(t), \Lambda_I^{cd}(t')]\rangle_{t_0} = \langle[F_I^{ab}(t - t' + t_0), \Lambda_I^{cd}(t_0)]\rangle_{t_0}$, this can be rewritten as

$$\langle F^{ab}\rangle_t - \langle F^{ab}\rangle_{t_0} = -\int_{-\infty}^{\infty} dt'\ \mathcal{X}^{abcd}(t - t')\frac{\partial\alpha_{cd}(t')}{\partial t'}\ , \tag{B.2.10}$$

where we defined the response function

$$\mathcal{X}^{abcd}(t) = \lim_{\epsilon\to 0^+} i\Theta(t)\langle[F_I^{ab}(t + t_0), \Lambda_I^{cd}(t_0)]\rangle_{t_0} e^{-\epsilon t}\ , \tag{B.2.11}$$

and where we also sent $t_0 \to -\infty$ in Eq. (B.2.10). Note that in Eq. (B.2.10) the Heaviside function $\Theta(t - t')$ allows us to extend the upper limit of the integral over t' to $+\infty$, while the presence of the factor $e^{-\epsilon(t-t')}$ allows us to send $t_0 \to -\infty$.

Next we perform a Fourier transform[2] and consider the frequency-dependent response function

$$\begin{aligned}\mathcal{X}^{abcd}(\omega) &= \int_{-\infty}^{\infty} dt\ \mathcal{X}^{abcd}(t)e^{i\omega t} \\ &= \lim_{\epsilon\to 0^+} i\int_0^{\infty} dt\ e^{i\omega_+ t}\langle[F_I^{ab}(t + t_0), \Lambda_I^{cd}(t_0)]\rangle_{t_0}\ ,\end{aligned} \tag{B.2.12}$$

where $\omega_+ = \omega + i\epsilon$. Now we note that

$$F_I^{ab}(t + t_0) = -i[\Lambda_I^{ab}(t + t_0), H_{CSMM}] = \hbar\frac{d\Lambda_I^{ab}(t + t_0)}{dt}\ , \tag{B.2.13}$$

where we used the equation of motion for $\Lambda_I^{ab}(t + t_0)$ in the interaction picture. Then an integration by parts with respect to t in the expression for $\mathcal{X}^{abcd}(\omega)$ yields a "strain-strain" form of the response function $\mathcal{X}^{abcd}(\omega)$ analogous to Eq. (3.5) of Ref. [92],

$$\mathcal{X}^{abcd}(\omega) = -i\hbar\langle[\Lambda^{ab}(t_0), \Lambda^{cd}(t_0)]\rangle_{t_0} + \lim_{\epsilon\to 0^+}\hbar\omega_+\int_0^{\infty} dt\ e^{i\omega_+ t}\langle[\Lambda^{ab}(t + t_0), \Lambda^{cd}(t_0)]\rangle_{t_0}\ . \tag{B.2.14}$$

[2]Our convention for Fourier transforms is $f(\omega) = \int_{-\infty}^{\infty} dt\ f(t)e^{i\omega t}$, $f(t) = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} f(\omega)e^{-i\omega t}$.

In the case where the unperturbed Hamiltonian has a unique ground state and a finite energy gap one finds that

$$\begin{aligned}\lim_{\omega\to 0}\mathcal{X}^{abcd}(\omega) &= -i\hbar\langle[\Lambda^{ab}(t_0),\Lambda^{cd}(t_0)]\rangle_{t_0} \\ &= -i\hbar\langle\psi_0|[\Lambda^{ab},\Lambda^{cd}]|\psi_0\rangle\ , \end{aligned} \tag{B.2.15}$$

i.e., the first term in Eq. (B.2.14) gives the full response at $\omega = 0$ [92]. These assumptions (unique ground state and finite energy gap) hold for the CSMM for any finite value of $\tilde{\omega}$, and so this formula for the response at $\omega = 0$ can be applied to the CSMM[3]. We note that this form of the response at $\omega = 0$ is what one obtains from a Hall viscosity calculation using adiabatic perturbation theory [22, 87, 91, 101].

Finally, we can complete the calculation of $\langle F^{ab}\rangle_t \equiv \langle\phi(t)|F^{ab}|\phi(t)\rangle$ to lowest order in time derivatives of $\alpha_{cd}(t)$. First, after a Fourier transformation (taking $t_0 \to -\infty$ in order to do the integration over t') we can write

$$\langle F^{ab}\rangle_t - \langle F^{ab}\rangle_{t_0} = \int_{-\infty}^{\infty}\frac{d\omega}{2\pi}\ i\omega\mathcal{X}^{abcd}(\omega)\alpha_{cd}(\omega)e^{-i\omega t}\ . \tag{B.2.16}$$

Next, we expand $\mathcal{X}^{abcd}(\omega)$ about $\omega = 0$ as

$$\mathcal{X}^{abcd}(\omega) = -i\hbar\langle\psi_0|[\Lambda^{ab},\Lambda^{cd}]|\psi_0\rangle + \dots \tag{B.2.17}$$

and invert the Fourier transformation to find

$$\langle F^{ab}\rangle_t - \langle F^{ab}\rangle_{t_0} = i\hbar\langle\psi_0|[\Lambda^{ab},\Lambda^{cd}]|\psi_0\rangle\frac{\partial\alpha_{cd}(t)}{\partial t} + \dots \tag{B.2.18}$$

For a system with an area A ($A = 2\pi\ell_B^2 mN$ for the CSMM with $\theta = \ell_B^2 m$) we then find that the Hall viscosity tensor is given by

$$\eta^{abcd}_{\text{CSMM}} = \frac{i\hbar}{A}\langle\psi_0|[\Lambda^{ab},\Lambda^{cd}]|\psi_0\rangle\ , \tag{B.2.19}$$

and this tensor encodes the linear response of the “generalized stress” $\frac{F^{ab}}{A}$ to the “rate of strain” given by $\frac{\partial\alpha_{cd}(t)}{\partial t}$.

[3] One should not confuse ω, the frequency appearing in the Fourier transform of the response function, with $\tilde{\omega}$, which sets the strength of the parabolic potential in the CSMM.

Appendix C

Supplement to Chapter 4

C.1 Equivariant Cohomology Intepretation of Gauged Wess-Zumino Actions

In this Appendix we review the connection between the theory of gauged WZ actions and equivariant cohomology. This allows us to give a concrete mathematical interpretation of the form of the gauged WZ actions for the boundary theories of BIQH and BTI states that we derived in Sec. 4.4 and Sec. 4.6 of this Chapter. Briefly, equivariant cohomology can be thought of as a generalization of de Rham cohomology to the case where a continuous group G acts on the manifold. In the cases of interest in this Chapter the group G is just the group $U(1)$ representing the charge-conservation symmetry of the SPT phases we study (i.e., the BIQH and BTI states), and this group acts on the target space of the NLSM via the rotations shown in Eq. (4.3.13). The connection between gauged WZ actions and equivariant cohomology has been explored in Refs. [39, 210, 211, 220]. The connection was first discussed explicitly by Witten in Ref. [39] for the case of two spacetime dimensions. Later, Figueroa-O'Farrill and Stanciu[210, 211] considered NLSMs with a generic target space and in any spacetime dimension, and they gave an explanation of the results of Ref. [41] in terms of equivariant cohomology. In addition, Wu[209] considered the equivalent mathematical problem of finding obstructions to the *equivariant extension* (to be defined below) of closed differential forms which are invariant under a group action. The result of these papers is that the problem of constructing a gauge-invariant WZ action is equivalent to the problem of constructing an equivariant extension of the volume form on the target manifold of the NLSM. We now give a brief review of equivariant cohomology and the connection to gauged WZ actions in the case where $G = U(1)$, and then we apply this knowledge to give a mathematical interpretation of the counterterms of Eq. (4.4.29) and Eq. (4.6.25) which appear in the gauged WZ actions constructed in this Chapter.

C.1.1 Equivariant cohomology

To introduce equivariant cohomology we first need to recall some basic facts about calculus on manifolds. For a D-dimensional manifold $\mathcal{M}$ a vector field $\underline{V}$ in the coordinate patch with coordinates $y = (y^1, \dots, y^D)$ can be expanded as

$$\underline{V} = V^a \frac{\partial}{\partial y^a} \ . \tag{C.1.1}$$

The partial derivatives $\frac{\partial}{\partial y^a}$ provide a basis for the tangent space $T_y\mathcal{M}$ of $\mathcal{M}$ at the point y, and a general vector field $\underline{V}$ is a section of the tangent bundle $T\mathcal{M}$ of $\mathcal{M}$. The differential forms dy^a provide a basis which is dual to the basis provided by $\frac{\partial}{\partial y^a}$, i.e., the dy^a form a basis for the cotangent space $T^*_y\mathcal{M}$ at the point y. A general differential p-form α is a section of the bundle whose fiber over the point y is $\bigwedge^p(T^*_y\mathcal{M})$, the p^{th} exterior power of $T^*_y\mathcal{M}$.

Now for any vector field $\underline{V}$ and p-form $\alpha = \frac{1}{p!}\alpha_{b_1\cdots b_p} dy^{b_1} \wedge \cdots \wedge dy^{b_p}$ we can define the operator $i_{\underline{V}}$, called *interior multiplication* by $\underline{V}$, by

$$i_{\underline{V}}\alpha = \frac{1}{(p-1)!} V^a \alpha_{ab_2\cdots b_p} dy^{b_2} \wedge \cdots \wedge dy^{b_p} \ . \tag{C.1.2}$$

So $i_{\underline{V}}$ takes a p-form to a $(p-1)$-form. For later use we also note that applying the interior multiplication twice gives zero, $i^2_{\underline{V}} = 0$, and that $i_{\underline{V}} f = 0$ for any function (zero form) on $\mathcal{M}$. The Lie derivative $\mathcal{L}_{\underline{V}}$ of any differential form α along the vector field $\underline{V}$ is then given by *Cartan's formula*,

$$\mathcal{L}_{\underline{V}}\alpha = d(i_{\underline{V}}\alpha) + i_{\underline{V}}(d\alpha) \ , \tag{C.1.3}$$

or simply

$$\mathcal{L}_{\underline{V}} = d i_{\underline{V}} + i_{\underline{V}} d \ , \tag{C.1.4}$$

in operator form.

We are now ready to introduce $U(1)$-equivariant cohomology over $\mathcal{M}$. To start, we pick some vector field $\underline{V}$ which generates a $U(1)$ action, or circle action, on the manifold. This can be understood concretely in terms of the flow generated by $\underline{V}$ as follows. First, recall that a vector field $\underline{V}$ generates a flow on the manifold via the set of differential equations

$$\frac{dy^a(t)}{dt} = V^a(y^1, \ldots, y^D) \ , \ a = 1, \ldots, D \ . \tag{C.1.5}$$

The condition that $\underline{V}$ generate a $U(1)$ action on the manifold means that this flow carries each point on $\mathcal{M}$ along a closed path, and each point takes the same amount of "time" t to return to its initial position. Now define the modified exterior derivative

$$\tilde{d} = d - i_{\underline{V}} \ . \tag{C.1.6}$$

Note that $\tilde{d}$ takes a p-form to a linear combination of a $(p+1)$-form and a $(p-1)$-form. If we compute the square of

$\tilde{d}$ then we find that

$$\tilde{d}^2 = -\mathcal{L}_{\underline{V}} \ , \tag{C.1.7}$$

which means that $\tilde{d}^2 = 0$ on the subspace of forms which have a vanishing Lie derivative along $\underline{V}$. It is therefore possible to define the cohomology of the operator $\tilde{d}$ in this subspace of differential forms in the same way that one defines the ordinary de Rham cohomology of the exterior derivative d.

Given this structure one can then try to understand what kinds of objects are closed under the action of $\tilde{d}$. From the definition of $\tilde{d}$ it is clear that a differential form of a definite degree will not, in general, be closed under the action of $\tilde{d}$. Instead, an equivariantly closed "form" α is actually a formal linear combination of differential forms of different degrees, i.e., a section of the bundle whose fiber over the point y is the exterior algebra $\bigwedge(T^*_y\mathcal{M}) = \bigoplus_{r=0}^{D} \bigwedge^r(T^*_y\mathcal{M})$. For the purposes of this Chapter we are interested in the case where α is a sum of a form of degree D (the highest possible degree form on the manifold), and several other forms whose parity (even or odd) is the same as that of the form of degree D. In this case we can expand α as

$$\alpha = \sum_{r=0}^{D'} \alpha^{(D-2r)} \ , \tag{C.1.8}$$

where $\alpha^{(D-2r)}$ is a differential form of degree $D-2r$ and

$$D' = \begin{cases} \frac{D}{2} & D = \text{ even} \\ \frac{D-1}{2} & D = \text{ odd} \ . \end{cases} \tag{C.1.9}$$

The condition $\tilde{d}\alpha = 0$ then implies that the forms $\alpha^{(D-2r)}$ obey the set of equations

$$i_{\underline{V}}\alpha^{(D-2r)} = d\alpha^{(D-2r-2)} \ , \ r = 0, \ldots, D'-1 \tag{C.1.10a}$$

$$i_{\underline{V}}\alpha^{(D-2D')} = 0 \ . \tag{C.1.10b}$$

In these equations the second line is trivially satisfied in the case that D is even, since in that case $D - 2D' = 0$ and so $\alpha^{(D-2D')}$ is just a function. The relation $d\alpha^{(D)} = 0$ is also trivially satisfied since $\alpha^{(D)}$ is a highest-degree form on $\mathcal{M}$, and so we have not included it in the set of equations for the forms that make up α. The form α constructed in this way is known as an *equivariant extension* of the form $\alpha^{(D)}$. We now move on and discuss the connection between these ideas and the theory of gauged WZ actions.

C.1.2 The connection to gauged WZ actions

To understand the connection between equivariant cohomology and gauged WZ actions, consider a general NLSM with D-dimensional target space $\mathcal{M}$ (a closed, compact manifold). We denote the NLSM field by $\boldsymbol{\phi} = (\phi^1, \dots, \phi^D)$, so $\boldsymbol{\phi}$ labels a point on $\mathcal{M}$. We formulate this NLSM on a spacetime manifold $\partial\mathcal{B}$ of dimension $D-1$, where $\mathcal{B}$ is an extended spacetime of dimension D. So the NLSM field $\boldsymbol{\phi}$ is a map from $\partial\mathcal{B}$ to $\mathcal{M}$. Finally, let $\alpha^{(D)}(\boldsymbol{\phi})$ be the volume form on the target space $\mathcal{M}$. Then a WZ term for this NLSM takes the form (we absorb any constant factors needed for consistency of the WZ term into the definition of $\alpha^{(D)}$)

$$S_{WZ}[\phi] = \int_{\mathcal{B}} \tilde{\boldsymbol{\phi}}^* \alpha^{(D)} , \tag{C.1.11}$$

where $\tilde{\boldsymbol{\phi}}$ is an extension of $\boldsymbol{\phi}$ into $\mathcal{B}$ and $\tilde{\boldsymbol{\phi}}^*\alpha^{(D)}$ again denotes the pullback of $\alpha^{(D)}$ to $\mathcal{B}$ via the map $\tilde{\boldsymbol{\phi}}$. In what follows we again omit the pullback symbols $\boldsymbol{\phi}^*$ and $\tilde{\boldsymbol{\phi}}^*$ for notational simplicity.

Now we suppose that the NLSM has a $U(1)$ symmetry and we attempt to probe this symmetry by coupling the system to the external field A. The transformation of the field $\boldsymbol{\phi}$ under the $U(1)$ symmetry is generated by a vector field $\underline{V}$, i.e., under an infinitesimal gauge transformation the field $\boldsymbol{\phi}$ transforms as

$$\phi^a \to \phi^a + \xi V^a , \tag{C.1.12}$$

where ξ is a small gauge transformation parameter. More generally, a differential p-form $\beta = \frac{1}{p!}\beta_{a_1\cdots a_p} d\phi^{a_1} \wedge \cdots \wedge d\phi^{a_p}$ on $\mathcal{M}$ transforms under a small gauge transformation as

$$\beta \to \beta + \mathcal{L}_{\xi\underline{V}}\beta , \tag{C.1.13}$$

where $\mathcal{L}_{\xi\underline{V}}$ is the Lie derivative along the "small" vector field $\xi\underline{V}$. We can now use this more general geometric formulation to try and gauge the WZ term. We should mention that in the case of a $U(1)$ symmetry it suffices to study the change in the action under infinitesimal gauge transformations since there is only one gauge transformation parameter ξ (as opposed to the non-Abelian case where there are several parameters ξ_J with J indexing the generators of the Lie group).

Under a small gauge transformation the WZ term transforms as

$$\begin{aligned}\delta_\xi S_{WZ}[\phi] &= \int_{\mathcal{B}} \mathcal{L}_{\xi\underline{V}}\alpha^{(D)} \\ &= \int_{\partial\mathcal{B}} \xi(i_{\underline{V}}\alpha^{(D)}) ,\end{aligned} \tag{C.1.14}$$

where we used the Lie derivative formula Eq. (C.1.13), the fact that $d\alpha^{(D)} = 0$, and the property $i_{\xi\underline{V}} = \xi i_{\underline{V}}$ of the interior multiplication. This change can be canceled by a term

$$S_{ct}^{(1)}[\phi, A] = \int_{\partial\mathcal{B}} A \wedge \alpha^{(D-2)} \ , \tag{C.1.15}$$

where $\alpha^{(D-2)}$ is some $(D-2)$-form, provided that $\alpha^{(D-2)}$ satisfies the equation

$$i_{\underline{V}}\alpha^{(D)} = d\alpha^{(D-2)} \ . \tag{C.1.16}$$

To see this, consider the change in $S_{ct}^{(1)}[\phi, A]$ when $A \to A + d\xi$. We find a term

$$\int_{\partial\mathcal{B}} d\xi \wedge \alpha^{(D-2)} = -\int_{\partial\mathcal{B}} \xi d\alpha^{(D-2)} \ , \tag{C.1.17}$$

where we performed an integration by parts and ignored boundary terms (since $\partial\mathcal{B}$ has no boundary). At this point the candidate gauged WZ action takes the form

$$S'_{WZ,gauged}[\phi, A] = S_{WZ}[\phi] + S_{ct}^{(1)}[\phi, A] \ . \tag{C.1.18}$$

Now under a small gauge transformation we find

$$\delta_\xi S'_{WZ,gauged}[\phi, A] = \int_{\partial\mathcal{B}} A \wedge (\mathcal{L}_{\xi\underline{V}}\alpha^{(D-2)}) \ , \tag{C.1.19}$$

which can be reduced to

$$\delta_\xi S'_{WZ,gauged}[\phi, A] = \int_{\partial\mathcal{B}} \xi F \wedge (i_{\underline{V}}\alpha^{(D-2)}) \ , \tag{C.1.20}$$

with the help of Eq. (C.1.16), the property $i_{\underline{V}}^2 = 0$, and an integration by parts. This change can then be canceled by a term

$$S_{ct}^{(2)}[\phi, A] = \int_{\partial\mathcal{B}} A \wedge F \wedge \alpha^{(D-4)} \ , \tag{C.1.21}$$

where $\alpha^{(D-4)}$ is some $(D-4)$-form that satisfies the equation

$$i_{\underline{V}}\alpha^{(D-2)} = d\alpha^{(D-4)} \ , \tag{C.1.22}$$

and so on.

Proceeding in this way we find that a gauge-invariant WZ term can be constructed if and only if there exist differential forms $\alpha^{(D-2r)}$, $r = 1, \dots, D'$, such that together with the volume form $\alpha^{(D)}$ they satisfy Eqs. (C.1.10). Thus, we find that the problem of constructing a gauge-invariant WZ action is exactly the same as the problem of constructing an equivariant extension of the volume form $\alpha^{(D)}$ on the target space manifold $\mathcal{M}$. We now use this information to re-interpret the gauged WZ actions for the boundary theories of the BIQH and BTI phases.

C.1.3 Application to BIQH and BTI boundary theories

In the BIQH and BTI cases the vector field $\underline{V}$ which generates the $U(1)$ gauge transformations is

$$\underline{V} = \sum_{\ell=1}^{m} \left(-n^{2\ell} \frac{\partial}{\partial n^{2\ell-1}} + n^{2\ell-1} \frac{\partial}{\partial n^{2\ell}} \right) . \tag{C.1.23}$$

Now the NLSMs which describe the boundary of the BIQH and BTI have target spaces S^{2m-1} and S^{2m}, respectively. We now consider the mathematical problem of constructing equivariant extensions of the volume forms ω_{2m-1} and ω_{2m} for these two manifolds. In the BTI case we will see that an equivariant extension of ω_{2m} exists, and we will give an explicit formula for it. On the other hand, in the BIQH case we will attempt to construct an equivariant extension of ω_{2m-1}, but we will find that it is not quite closed under the action of $\tilde{d} = d - i_{\underline{V}}$. This gives a mathematical interpretation of the $U(1)$ anomaly that we found for the boundary theory of the BIQH phase.

We start with the BTI case. Recall from our study of the boundary theory of the BTI that the construction of the gauged WZ action involved the forms $\Phi^{(r)}$ from Eq. (4.6.3). If we apply interior multiplication by $\underline{V}$ to these forms we find

$$i_{\underline{V}} \Phi^{(r)} = (m - r) n_{2m+1} dn_{2m+1} \wedge \Phi^{(r+1)} , \tag{C.1.24}$$

which bears a close resemblance to Eq. (4.6.4). In addition, for the volume form ω_{2m} we have

$$i_{\underline{V}} \omega_{2m} = \frac{1}{(m-1)!} d \left(n_{2m+1} \Phi^{(1)} \right) . \tag{C.1.25}$$

We now use these relations to construct an equivariant extension of ω_{2m}, i.e., a solution of Eqs. (C.1.10) with $\alpha^{(D)} = \omega_{2m}$ (so $D = 2m$). To start we need a form $\alpha^{(2m-2)}$ which satisfies

$$i_{\underline{V}} \omega_{2m} = d\alpha^{(2m-2)} , \tag{C.1.26}$$

and from Eq. (C.1.25) the answer is obviously

$$\alpha^{(2m-2)} = \frac{1}{(m-1)!} n_{2m+1} \Phi^{(1)} . \tag{C.1.27}$$

Next we need a form $\alpha^{(2m-4)}$ such that

$$i_{\underline{V}}\alpha^{(2m-2)} = d\alpha^{(2m-4)} \ , \tag{C.1.28}$$

and Eq. (C.1.24) tells us exactly how to find such a form. Proceeding in this way we eventually find that an equivariant extension of ω_{2m} exists and is given explicitly by

$$\tilde{\omega}_{2m} = \omega_{2m} + \sum_{r=1}^{m} \frac{1}{(m-r)!(2r-1)!!}(n_{2m+1})^{2r-1}\Phi^{(r)} \ . \tag{C.1.29}$$

The terms appearing in the equivariantly closed form $\tilde{\omega}_{2m}$ are exactly the same as the terms which appear multiplying the factors $A \wedge F^{r-1}$ in the counterterms of Eq. (4.6.25) for the gauged action of the BTI boundary. So our construction of a gauged WZ action for the BTI boundary is equivalent to the construction of an equivariant extension of the volume form ω_{2m} on S^{2m}.

Moving on to the BIQH phase, we recall that in the BIQH case the construction of the gauged WZ action involved the forms $\Omega^{(r)}$ defined in Eq. (4.4.13). Applying the interior multiplication by $\underline{V}$ to these forms gives

$$i_{\underline{V}}\Omega^{(r)} = \frac{1}{2}d\Omega^{(r+1)} \ , \tag{C.1.30}$$

which bears a close resemblance to Eq. (4.4.14). We also saw that the volume form ω_{2m-1} for S^{2m-1} could be written in terms of the $\Omega^{(r)}$ as $\omega_{2m-1} = \frac{1}{(m-1)!}\Omega^{(0)}$. Using this fact, and Eq. (C.1.30), we can then attempt to construct an equivariant extension of ω_{2m-1}, using the same procedure as in the BTI case. In this way we find a candidate for an equivariant extension of ω_{2m-1}, which is given explicitly by

$$\tilde{\omega}_{2m-1} = \omega_{2m-1} + \frac{1}{(m-1)!}\sum_{r=1}^{m-1}\frac{1}{2^r}\Omega^{(r)} \ . \tag{C.1.31}$$

However, this object is not quite closed under the action of $\tilde{d} = d - i_{\underline{V}}$, and instead we find that

$$\tilde{d}\tilde{\omega}_{2m-1} = -\frac{1}{(m-1)!}\frac{1}{2^{m-1}} \ . \tag{C.1.32}$$

In fact, what has happened is that the second line of Eqs. (C.1.10) fails to hold in this case. This failure of $\tilde{\omega}_{2m-1}$ to be equivariantly closed is the mathematical reason for why the BIQH boundary action is not gauge-invariant, but instead has a pertubative anomaly in the $U(1)$ symmetry.

It turns out that there is a simple mathematical explanation for why an equivariant extension of ω_{2m-1} does not exist in this case [1]. For the $U(1)$ symmetry considered in this Chapter (see Eq. (4.3.13)) the action of the group

[1] This explanation was pointed out to us by Michael Stone and we thank him for sharing it with us.

$U(1)$ on S^{2m-1} is free, i.e., only the identity element of $U(1)$ leaves all the points in S^{2m-1} fixed. In this case the $U(1)$-equivariant cohomology of S^{2m-1} is equal to the ordinary de Rham cohomology of the quotient manifold $S^{2m-1}/U(1)$ (see, for example, Ref. [208]). Now for the specific $U(1)$ symmetry we have chosen the quotient is just $S^{2m-1}/U(1) = C\mathbb{P}^{m-1}$, and we know that the cohomology ring of $C\mathbb{P}^{m-1}$ is generated by the Kähler two-form K (which we will meet in Appendix C.2). This means that only the even-dimensional cohomology groups of $C\mathbb{P}^{m-1}$ are non-trivial. On the other hand, the volume form of S^{2m-1} is a $(2m-1)$-form, i.e., a form of *odd* degree. Since the $U(1)$-equivariant cohomology of S^{2m-1} is equivalent to the ordinary cohomology of $C\mathbb{P}^{m-1}$, we conclude that an equivariant extension of ω_{2m-1} does not exist for this $U(1)$ symmetry (if such an extension did exist, then it would imply the existence of a non-trivial closed form of odd degree on $C\mathbb{P}^{m-1}$, but no such form exists).

C.2 Chern character on $C\mathbb{P}^m$

In this Appendix we compute the integral

$$\int_X \left(\frac{F}{2\pi}\right)^m \tag{C.2.1}$$

for the specific case of $X = C\mathbb{P}^m$ (complex projective space with m complex dimensions). When the field strength F satisfies the Dirac quantization condition of Eq. (4.5.5) in Sec. 4.5 we find that the integral can be equal to one. This answer is already well-known, but it provides a nice example of the need for the peculiar quantization of the CS level on generic manifolds, as we discussed in Sec. 4.5.

To compute the integral in Eq. (C.2.1) we are going to need some background information about the complex projective space $C\mathbb{P}^m$. We choose to follow the discussion in Ref. [56]. Note that in this section we depart from previous notation and use an overline $\bar{z}$, and not a star, to denote the complex conjugate of a complex number z. For $C\mathbb{P}^m$ the second Betti number is $b_2 = \dim[H_2(X,\mathbb{R})] = 1$, meaning that $C\mathbb{P}^m$ has a single non-trivial two-cycle. This two-cycle, which we call $\mathcal{C}$, is essentially a copy of $C\mathbb{P}^1$. To understand this two-cycle, and the element of $H^2(X,\mathbb{R})$ which is dual to it, first introduce the Kähler form K on $C\mathbb{P}^m$,

$$K = \frac{i}{2} g_{ab}\, dz^a \wedge d\bar{z}^b \ , \tag{C.2.2}$$

where

$$g_{ab} = \frac{1}{\mathcal{D}^2}\left[\mathcal{D}\delta_{ab} - \bar{z}_a z_b\right] \ , \tag{C.2.3}$$

and

$$\mathcal{D} = 1 + \sum_{c=1}^{m} z_c \bar{z}_c \ . \tag{C.2.4}$$

Here z_a, $a = 1, \ldots, m$, are complex coordinates which each take values on the whole complex plane $\mathbb{C}$. The indices of z_a can be raised and lowered with δ_{ab} and δ^{ab}, and as usual there is an implied sum over any index which appears once in a lower position and once in an upper position in any expression. The quantity g_{ab} is known as the Fubini-Study metric and it satisfies $\bar{g}_{ab} = g_{ba}$. In addition we have $dK = 0$, so K is closed. That K is closed follows immediately from the fact that it can be written as

$$K = \frac{i}{2}\partial\overline{\partial}\ln(\mathcal{D}) \ , \tag{C.2.5}$$

where $\partial \equiv \partial_{z^a} dz^a$, $\overline{\partial} \equiv \partial_{\bar{z}^a} d\bar{z}^a$ are the Dolbeault operators (on any Kähler manifold one has $K = \frac{i}{2}\partial\overline{\partial}\rho$, where the function ρ is called the Kähler potential). Since the exterior derivative decomposes as $d = \partial + \overline{\partial}$, and since the Dolbeault operators satisfy $\partial^2 = \overline{\partial}^2 = \{\partial, \overline{\partial}\} = 0$, we immediately see that $dK = 0$. Hence, the Kähler form is closed. However, it is not exact, and we will use it in order to write down non-trivial configurations of F on $C\mathbb{P}^m$.

The Kähler form K is a representative of the non-trivial element of $H^2(X, \mathbb{R})$. In the coordinate patch that we have chosen (in which K takes the form shown in Eq. (C.2.2)) we can take the non-trivial two-cycle $\mathcal{C}$ to be any *one* of the m complex planes whose coordinates are z_a. For example let us take $\mathcal{C}$ to be the z_1 plane. In that plane (with all other $z_a = 0$) we have

$$K \to \left(\frac{i}{2}\right) \frac{dz^1 \wedge d\bar{z}^1}{(1 + z_1 \bar{z}_1)^2} \ . \tag{C.2.6}$$

If we introduce the real coordinates x_1 and x_2 by $z_1 = x_1 + ix_2$ then we have $dz^1 \wedge d\bar{z}^1 = -2idx^1 \wedge dx^2$, and integrating K over the (x_1, x_2) plane gives

$$\int_{\mathcal{C}} K = \int \frac{dx^1 \wedge dx^2}{(1 + x_1^2 + x_2^2)^2} = \pi \ . \tag{C.2.7}$$

We learn from this that a normalized form with unit flux through $\mathcal{C}$ is $\frac{K}{\pi}$, so we should set $\frac{F}{2\pi} = \frac{K}{\pi}$ or just

$$F = 2K \ , \tag{C.2.8}$$

in order to satisfy the Dirac quantization condition of Eq. (4.5.5).

Now in order to compute the integral in Eq. (C.2.1) we need to do the integral

$$\int_{C\mathbb{P}^m} K^m \ , \tag{C.2.9}$$

so we need to compute the wedge product of K with itself m times. We have

$$K^m = \left(\frac{i}{2}\right)^m \frac{1}{\mathcal{D}^{2m}} dz^{a_1} \wedge d\bar{z}^{b_1} \wedge \cdots \wedge dz^{a_m} \wedge d\bar{z}^{b_m} \prod_{r=1}^{m} \left[\mathcal{D}\delta_{a_r b_r} - \bar{z}_{a_r} z_{b_r}\right] \ . \tag{C.2.10}$$

To simplify this, first note that

$$dz^{a_1} \wedge d\bar{z}^{b_1} \wedge \cdots \wedge dz^{a_m} \wedge d\bar{z}^{b_m} = \epsilon^{a_1 \cdots a_m} \epsilon^{b_1 \cdots b_m} dz^1 \wedge d\bar{z}^1 \wedge \cdots \wedge dz^m \wedge d\bar{z}^m \ . \tag{C.2.11}$$

Now we have to contract $\epsilon^{a_1 \cdots a_m} \epsilon^{b_1 \cdots b_m}$ with the product

$$\prod_{r=1}^{m} \left[\mathcal{D} \delta_{a_r b_r} - \bar{z}_{a_r} z_{b_r} \right] \ . \tag{C.2.12}$$

When expanded out this product contains 2^m terms. However, most of these terms contain two or more factors of $\bar{z}_{a_r} z_{b_r}$, for example a term might contain two of them such as $\bar{z}_{a_1} z_{b_1} \bar{z}_{a_2} z_{b_2}$. All such terms with two or more factors of $\bar{z}_{a_r} z_{b_r}$ will vanish when contracted with $\epsilon^{a_1 \cdots a_m} \epsilon^{b_1 \cdots b_m}$ because of the anti-symmetry of the Levi-Civita symbol, so we only have to worry about terms with zero or one factor of $\bar{z}_{a_r} z_{b_r}$. The term with no factors of $\bar{z}_{a_r} z_{b_r}$ is

$$\mathcal{D}^m \prod_{r=1}^{m} \delta_{a_r b_r} \ , \tag{C.2.13}$$

and we have

$$\epsilon^{a_1 \cdots a_m} \epsilon^{b_1 \cdots b_m} \mathcal{D}^m \prod_{r=1}^{m} \delta_{a_r b_r} = m! \mathcal{D}^m \ . \tag{C.2.14}$$

Then there are m terms which each have a single factor of $\bar{z}_{a_r} z_{b_r}$. The first such term is

$$- \bar{z}_{a_1} z_{b_1} \mathcal{D}^{m-1} \prod_{r=2}^{m} \delta_{a_r b_r} \ , \tag{C.2.15}$$

and we find

$$- \epsilon^{a_1 \cdots a_m} \epsilon^{b_1 \cdots b_m} \bar{z}_{a_1} z_{b_1} \mathcal{D}^{m-1} \prod_{r=2}^{m} \delta_{a_r b_r} = -\mathcal{D}^{m-1} (m-1)! \sum_{c=1}^{m} z_c \bar{z}_c \ . \tag{C.2.16}$$

So all together we find that (recalling that there are m terms with one factor of $\bar{z}_{a_r} z_{b_r}$ and they all give an identical contribution)

$$\epsilon^{a_1 \cdots a_m} \epsilon^{b_1 \cdots b_m} \prod_{r=1}^{m} \left[\mathcal{D} \delta_{a_r b_r} - \bar{z}_{a_r} z_{b_r} \right] = m! \mathcal{D}^{m-1} \ , \tag{C.2.17}$$

where we used $(\mathcal{D} - \sum_{c=1}^{m} z_c \bar{z}_c) = 1$. We finally obtain

$$K^m = m! \left(\frac{i}{2} \right)^m \frac{dz^1 \wedge d\bar{z}^1 \wedge \cdots \wedge dz^m \wedge d\bar{z}^m}{\mathcal{D}^{m+1}} \ . \tag{C.2.18}$$

To do the integral over $C\mathbb{P}^m$ we now introduce $2m$ real coordinates x_j, $j = 1, \ldots, 2m$, defined by $z_j = x_{2j-1} +$

ix_{2j}. Let $r^2 = \sum_{j=1}^{2m} x_j^2$. The integral becomes

$$\begin{aligned}
\int_{C\mathbb{P}^m} K^m &= m! \int d^{2m}x \frac{1}{(1+r^2)^{m+1}} \\
&= m!\, \mathcal{A}_{2m-1} \int_0^\infty dr\, \frac{r^{2m-1}}{(1+r^2)^{m+1}} \\
&= m!\, \mathcal{A}_{2m-1}\, \frac{1}{2m} \\
&= \pi^m\ ,
\end{aligned} \tag{C.2.19}$$

where we used spherical coordinates on $\mathbb{R}^{2m}$ to do the integral. Setting $F = 2K$, we then find that

$$\int_{C\mathbb{P}^m} \left(\frac{F}{2\pi}\right)^m = 1\ . \tag{C.2.20}$$

C.3 From BIQH to BTI states via dimensional reduction

In this Appendix we discuss a dimensional reduction procedure which allows one to generate a BTI state in $2m-2$ dimensions from a BIQH state in $2m-1$ dimensions. The procedure is carried out at the level of the effective action $S_{eff}[A]$ and is similar, but not equivalent to, the procedure used in Ref. [8] to obtain the time-reversal invariant topological insulator in four dimensions from an Integer Quantum Hall state of fermions in five dimensions.

To start, we imagine separately gauging the $U(1)$ symmetry associated with each species of "boson" b_ℓ ($\ell = 1, \dots, m$) in the NLSM description of the BIQH state in $2m-1$ dimensions. That is, we consider an $O(2m)$ NLSM describing a BIQH state, and we study this state with a $U(1)^m$ symmetry which acts on the bosons as

$$b_\ell \to e^{i\xi_\ell} b_\ell\ ,\ \ell = 1, \dots, m\ , \tag{C.3.1}$$

where ξ_ℓ are a set of m independent gauge transformation parameters. We then couple this system to m $U(1)$ gauge fields $A_\mu^{(\ell)}$.

In this Chapter we have not calculated the response of the $O(2m)$ NLSM when this $U(1)^m$ subgroup is gauged. However, from our results in this Chapter we can make an argument for what the general form should be. The effective response action $S_{eff}[A^{(1)}, \dots, A^{(m)}]$ should have at least two properties: (i) it should reduce to a CS response with level $N_{2m-1} = m!$ for the gauge field A if we set $A^{(\ell)} = A\ \forall \ell$, and (ii) it should be invariant under any permutation of the labels ℓ of the different gauge fields. The second property follows from the fact that the action for the $O(2m)$ NLSM is invariant under any permutation of the labels ℓ of the bosons b_ℓ. This fact is not completely obvious, and so we prove it now.

The $O(2m)$ NLSM with theta term or WZ term is invariant under the action of the *alternating* group A_{2m} of even signature permutations of the labels $a = 1, \dots, 2m$ of the components n_a of the NLSM field $\mathbf{n}$. Now the permutations of the labels $\ell = 1, \dots, m$ of the bosons b_ℓ consist of two transpositions in the *symmetric* group S_{2m}. This is because a permutation which swaps ℓ with ℓ' must swap $n_{2\ell-1}$ with $n_{2\ell'-1}$ and $n_{2\ell}$ with $n_{2\ell'}$. Since the signature of a permutation σ in the symmetric group is given by $\text{sgn}(\sigma) = (-1)^{N_T}$, with N_T the number of transpositions in σ, it immediately follows that the permutations of the boson labels ℓ are contained within the group A_{2m}. This proves property (ii).

Using properties (i) and (ii) we can now argue that the response action for a gauged $U(1)^m$ symmetry must take the form

$$S_{eff}[A^{(1)}, \dots, A^{(m)}] = \frac{k}{(2\pi)^{m-1} m!} \int_{\mathcal{M}} \left(A^{(1)} \wedge dA^{(2)} \wedge \cdots \wedge dA^{(m)} + \text{permutations} \right) . \tag{C.3.2}$$

If $\mathcal{M}$ has no boundary then we can integrate by parts and write this simply as

$$S_{eff}[A^{(1)}, \dots, A^{(m)}] = \frac{k}{(2\pi)^{m-1}} \int_{\mathcal{M}} A^{(m)} \wedge dA^{(1)} \wedge \cdots \wedge dA^{(m-1)} , \tag{C.3.3}$$

or we could choose some other ordering of the gauge fields $A^{(\ell)}$. We now describe the dimensional reduction procedure which allows one to derive the electromagnetic response for the BTI state in $2m-2$ dimensions from this response action for the BIQH state in $2m-1$ dimensions. For concreteness we work on flat spacetime with spatial coordinates x^j, $j = 1, \dots, 2m-2$.

To obtain the BTI state from the higher dimensional BIQH state we first compactify the space by wrapping the x^{2m-2} direction into a circle, which turns the space $\mathbb{R}^{2m-2}$ into the "cylinder" $\mathbb{R}^{2m-3} \times S^1$. We then thread a π flux of the gauge field $A^{(m)}$ through the hole in the cylinder, and finally shrink the circumference of the cylinder to zero. This leaves us with a response action for a phase in $2m-2$ spacetime dimensions. Mathematically, this procedure assumes the following configuration of gauge fields $A^{(\ell)}$: (i) $A^{(\ell)}$ $\ell = 1, \dots, m-1$, are independent of x^{2m-2} and have their last component $A^{(\ell)}_{\mu=2m-2}$ equal to zero, (ii) the components $A^{(m)}_\mu$, $\mu = 0, \dots, 2m-3$, of the m^{th} gauge field are equal to zero, and (iii) the last component of $A^{(m)}$ satisfies $\int dx^{2m-2}\, A^{(m)}_{2m-2} = \pi$.

Under these assumptions the effective action for the BIQH phase with gauged $U(1)^m$ symmetry reduces as

$$S_{eff}[A^{(1)}, \dots, A^{(m)}] \to \frac{\pi k}{(2\pi)^{m-1}} \int_{\mathbb{R}^{2m-3,1}} dA^{(1)} \wedge \cdots \wedge dA^{(m-1)} . \tag{C.3.4}$$

If we now break the remaining $U(1)^{m-1}$ symmetry of this phase down to $U(1)$ by setting $A^{(\ell)} = A$ for $\ell = 1, \dots, m-$

1 then we obtain (with $F = dA$)

$$S_{eff}[A^{(1)}, \dots, A^{(m)}] \to \frac{\pi k}{(2\pi)^{m-1}} \int_{\mathbb{R}^{2m-3,1}} F^{m-1} \ , \tag{C.3.5}$$

which is a response action of the form of Eq. (4.2.2) with response parameter $\Theta_{m-1} = \pi(m-1)!k$. This is exactly the response action for a BTI phase in $2m-2$ dimensions, so the dimensional reduction procedure described here does allow one to obtain the BTI response action from the BIQH response action in one higher dimension.

The main difference between the dimensional reduction procedure shown here and the procedure used in Ref. [8] is that in our procedure we only thread π flux of *one* flavor ℓ of gauge field $A^{(\ell)}$ through hole in the cylinder. On the other hand, the procedure in Ref. [8] (in which there is only a single $U(1)$ gauge field A) is equivalent to threading π flux of *all* the gauge fields $A^{(\ell)}$. This second method *does not* give the proper quantization of the parameter Θ_{m-1} for the BTI phase in one lower dimension. The answer turns out to be too large by a factor of m. The physical reason for the different dimensional reduction procedure needed to go from BIQH to BTI states can be seen from our alternative calculation in Sec. 4.4 of the BIQH response. There we showed that threading 2π flux of the external gauge field A generates a vortex in all m bosons b_ℓ, and so it creates m excitations. This explains why the more familiar dimensional reduction procedure of threading π flux for A gives an answer which is m times too large.

C.4 Dimensional reduction formulas for theta terms in nonlinear sigma models

In this section we derive a general dimensional reduction formula for theta terms of $O(D+2)$ NLSMs in $D+1$ dimensions. The formula shows how the theta term of the NLSM can reduce to a theta term for a lower-dimensional NLSM when evaluated on a "defect configuration" of the NLSM field. The formula we derive applies to any spacetime dimension $D+1$, and defects of any codimension, with the simplest cases being vortices and hedgehog defects. The physical content of the dimensional reduction formula can be summarized in the following way: a topological defect of (spatial) co-dimension $(q+1)$ in an $O(D+2)$ NLSM with theta term can trap an $O(D-q+1)$ NLSM with theta term in its core. The theta angle of the lower-dimensional NLSM is related to that of the original NLSM in a simple way which we calculate below. We use a special case of this formula in the last subsection of Sec. 4.4 to study vortices in the NLSM description of the BIQH state, but the general result presented in this Appendix should be very useful for working with these models. Dimensional reduction of topological terms in NLSMs was also considered in Appendix C of Ref. [188], but to the best of our knowledge the general formula presented in this Appendix has not appeared before in the literature. Finally, we also remark that the formula presented here can also be used when WZ terms are present, as the form of the WZ term is similar to the form of the theta term.

To start, recall the theta term

$$S_\theta[\mathbf{n}] = \frac{\theta}{\mathcal{A}_{D+1}} \int_{\mathbb{R}^{D,1}} \mathbf{n}^* \omega_{D+1} \ , \tag{C.4.1}$$

for a NLSM with field $\mathbf{n}(t,\mathbf{x})$ in $D+1$ spacetime dimensions, where t represents the time, and $\mathbf{x} = (x^1,\dots,x^D)$ represents the spatial coordinates. Here ω_{D+1} is the volume form for the sphere S^{D+1} which was introduced in Eq. (4.3.6). The integral is over $(D+1)$-dimensional Minkowski spacetime $\mathbb{R}^{D,1}$. To describe the defect configurations considered here, we first decompose the total spacetime as

$$\mathbb{R}^{D,1} = \mathbb{R}^{q+1} \times \mathbb{R}^{D-(q+1),1} \ , \tag{C.4.2}$$

and we further decompose the first factor into $(q+1)$-dimensional spherical coordinates as

$$\mathbb{R}^{q+1} = [0,\infty) \times S^q \ . \tag{C.4.3}$$

Here q is a positive integer which is going to be related to the codimension of the defect in the NLSM field.

We introduce coordinates $r \in [0,\infty)$ and $\mathbf{s} = (s^1,\dots,s^q)$ to parametrize $\mathbb{R}^{q+1} = [0,\infty) \times S^q$. The precise nature of the coordinates $\mathbf{s}$ for S^q will not be important to us here. We also use t and $\mathbf{y} = (y^1,\dots,y^{D-(q+1)})$ to denote the remaining coordinates on $\mathbb{R}^{D-(q+1),1}$. The defect configuration we consider takes the form

$$\mathbf{n}(t,\mathbf{x}) = \{\sin(f(r))\mathbf{N}(t,\mathbf{y}), \cos(f(r))\mathbf{m}(\mathbf{s})\} \ , \tag{C.4.4}$$

where $\mathbf{N}$ is a $(D-q+1)$-component unit vector which depends only on the coordinates $(t,\mathbf{y})$ for $\mathbb{R}^{D-(q+1),1}$, $\mathbf{m}$ is a $(q+1)$-component unit vector which depends only on the coordinates $\mathbf{s}$ for S^q, and where $f(r)$ is a function obeying the boundary conditions

$$f(0) = \frac{\pi}{2} \tag{C.4.5}$$

$$\lim_{r\to\infty} f(r) = 0 \ . \tag{C.4.6}$$

Physically, this form of $\mathbf{n}$ describes a defect of spatial codimension $q+1$ in which the field $\mathbf{m}$ takes on a non-trivial configuration on the sphere S^q. The field $\mathbf{N}$ then describes a lower-dimensional NLSM which lives in the core of this defect, and the core size is controlled by the profile of the function $f(r)$. The non-triviality of the configuration of $\mathbf{m}$

is captured by the winding number n_q of $\mathbf{m}$ on S^q,

$$n_q = \frac{1}{\mathcal{A}_q}\int_{S^q} \mathbf{m}^*\omega_q \ . \tag{C.4.7}$$

After some algebra one can show that the pullback $\mathbf{n}^*\omega_{D+1}$ of the volume form for the original NLSM field $\mathbf{n}$ will reduce on this configuration as

$$\mathbf{n}^*\omega_{D+1} \to (-1)^{(D-q)(q+1)+1}[\sin(f(r))]^{D-q}[\cos(f(r))]^q f'(r)dr \wedge \mathbf{m}^*\omega_q \wedge \mathbf{N}^*\omega_{D-q} \ . \tag{C.4.8}$$

This formula can be derived from the formula for $\mathbf{n}^*\omega_{D+1}$ by using the fact that wedge products of the differential of any coordinate with itself will vanish. This fact strongly constrains the terms which survive in $\mathbf{n}^*\omega_{D+1}$ once one assumes that $\mathbf{n}$ is in the defect configuration of Eq. (C.4.4). Now we just need to do the integrals over the radial direction (parameterized by r) and the sphere S^q to find the reduced theta term for $\mathbf{N}$. For the radial integral we have

$$\begin{aligned} I_r &\equiv -\int_0^\infty dr\ [\sin(f(r))]^{D-q}[\cos(f(r))]^q f'(r) \\ &= \int_0^{\frac{\pi}{2}} df\ [\sin(f)]^{D-q}[\cos(f)]^q \\ &= \frac{\Gamma(\frac{D-q+1}{2})\Gamma(\frac{q+1}{2})}{2\Gamma(\frac{D}{2}+1)} \ . \end{aligned} \tag{C.4.9}$$

Combining this with Eq. (C.4.7) for the winding of the defect in the $\mathbf{m}$ field, we find that the theta term of Eq. (C.4.1) for $\mathbf{n}$ reduces as

$$\begin{aligned} S_\theta[\mathbf{n}] &\to (-1)^{(D-q)(q+1)}\frac{\theta I_r}{\mathcal{A}_{D+1}}\int_{S^q}\mathbf{m}^*\omega_q\int_{\mathbb{R}^{D-(q+1),1}}\mathbf{N}^*\omega_{D-q} \\ &= \frac{\theta_{eff}}{\mathcal{A}_{D-q}}\int_{\mathbb{R}^{D-(q+1),1}}\mathbf{N}^*\omega_{D-q} \ , \end{aligned} \tag{C.4.10}$$

where the effective theta angle for the lower-dimensional NLSM is

$$\theta_{eff} = (-1)^{(D-q)(q+1)} n_q\,\theta \ , \tag{C.4.11}$$

and where we used the formula

$$\frac{\Gamma(\frac{D-q+1}{2})\Gamma(\frac{q+1}{2})}{2\Gamma(\frac{D}{2}+1)}\frac{\mathcal{A}_q}{\mathcal{A}_{D+1}} = \frac{1}{\mathcal{A}_{D-q}} \ . \tag{C.4.12}$$

So we see that on this defect configuration the original theta term for $\mathbf{n}$ has reduced to a theta term for the field $\mathbf{N}$ which lives in the core of the defect. In addition, from Eq. (C.4.11) we see that the theta angle θ_{eff} for this lower-dimensional NLSM is simply related to the original theta angle by a sign factor $(-1)^{(D-q)(q+1)}$ and by multiplication by the winding number n_q of the defect in $\mathbf{m}$.

C.5 Electromagnetic Response of $O(2)$ NLSM in one spacetime dimension

In this Appendix we derive the electromagnetic response of the $O(2)$ NLSM with theta term, which represents an analog of the BIQH state in 1 spacetime dimension. In the last subsection of Sec. 4.4 we presented an alternative derivation of the electromagnetic response of the $O(2m)$ NLSM with theta term at $\theta = 2\pi k$ in $2m-1$ dimensions, in which we were able to relate the level N_{2m-1} of the CS term in the response for the $O(2m)$ NLSM to the level N_1 for the response of the $O(2)$ NLSM at $\theta = 2\pi k$. Specifically, we found that the two levels were related as

$$N_{2m-1} = (m!)N_1 \ . \tag{C.5.1}$$

We now derive the formula

$$N_1 = -k \ , \tag{C.5.2}$$

for the $O(2)$ NLSM with $\theta = 2\pi k$, which we then use to complete our alternative derivation at the end of Sec. 4.4 of the formula $N_{2m-1} = -(m!)k$ for the CS response of the $O(2m)$ NLSM with $\theta = 2\pi k$.

We begin the derivation by parameterizing the $O(2)$ field as $\mathbf{n} = \{\cos(\varphi), \sin(\varphi)\}$ or as $b_1 = e^{i\varphi}$ in terms of the boson $b_1 = n_1 + in_2$. In terms of the angular variable φ the action for the $O(2)$ NLSM with theta term takes the form

$$S[\mathbf{n}] = \int_0^T dt \ \left\{ \frac{1}{2g}(\partial_t\varphi)^2 + \frac{\theta}{2\pi}\partial_t\varphi \right\} \ . \tag{C.5.3}$$

Here we have made the calculation as concrete as possible by considering a finite time interval $[0, T)$, and we assume periodic boundary conditions for the boson b_1 in the time direction. This leaves open the possibility that φ can wind around the time direction, i.e., we can have configurations in which $\varphi(t+T) = \varphi(t) + 2\pi n$ for an integer n. As in the higher-dimensional cases, we will be interested in the limit $g \to \infty$. In this one-dimensional case this limit just projects onto the ground state (or states) of this quantum mechanical system (in higher dimensions $g \to \infty$ corresponds to the disordered phase of the model).

The $U(1)$ symmetry which acts on b_1 as $b_1 \to e^{i\xi}b_1$ then acts on φ as

$$U(1): \ \varphi \to \varphi + \xi \ . \tag{C.5.4}$$

We would now like to couple φ to a $U(1)$ gauge field $A = A_t dt$. For the boundary conditions we are considering we can write A_t in the general form

$$A_t = \overline{A}_t + \delta A_t \ , \tag{C.5.5}$$

where $\overline{A}_t = \frac{1}{T}\int_0^T dt\ A_t$ and $\int_0^T dt\ \delta A_t = 0$. This is equivalent to the statement that the closed form A can be written as an exact part plus a piece which has a non-vanishing integral around the non-trivial one-cycle in the time direction, since we assumed periodic boundary conditions in time. We can always remove the exact part δA_t from A_t by a small $U(1)$ gauge transformation $\varphi \to \varphi + \xi$, $A \to A + d\xi$ with $\int d\xi = 0$. Therefore we will just work with the constant part $\overline{A}_t$ in what follows.

The gauged $O(2)$ NLSM action is obtained by the standard minimal coupling procedure,

$$S_{gauged}[\mathbf{n}, A] = \int_0^T dt\ \left\{ \frac{1}{2g}(\partial_t \varphi - \overline{A}_t)^2 + \frac{\theta}{2\pi}(\partial_t\varphi - \overline{A}_t) \right\} \ , \tag{C.5.6}$$

however, there is one subtle point here. This action is invariant under small and large $U(1)$ gauge transformations, where by a large $U(1)$ gauge transformation we mean a transformation in which $\int d\xi \neq 0$. Now if we only cared about invariance under small $U(1)$ gauge transformations, we could just as well have used the action

$$S'_{gauged}[\mathbf{n}, A] = \int_0^T dt\ \left\{ \frac{1}{2g}(\partial_t \varphi - \overline{A}_t)^2 + \frac{\theta}{2\pi}\partial_t\varphi \right\} \ , \tag{C.5.7}$$

which *does not* involve minimal coupling inside the theta term. This form is more relevant in cases in which one is interested in enforcing certain discrete symmetries *at the expense* of large $U(1)$ gauge invariance, as could be the case in the investigation of global anomalies in discrete symmetries of this theory at $\theta = \pi$ (compare with the discussion for fermionic systems in one dimension in Ref. [76]). This could be relevant for studies of the boundary states of SPT phases in two spacetime dimensions. In our case, however, we are interested in the $O(2m)$ NLSM in $2m-1$ dimensions as a low-energy description of a bosonic lattice model which can be coupled to a *compact* $U(1)$ gauge field, and so we gauge the theory in such a way as to preserve this large $U(1)$ gauge invariance. With these remarks in mind, we now proceed with the computation.

From Eq. (C.5.6) the momentum conjugate to φ is $p = \frac{1}{g}(\partial_t\varphi - \overline{A}_t) + \frac{\theta}{2\pi}$, and the Hamiltonian is

$$H = \frac{g}{2}\left(p - \frac{\theta}{2\pi}\right)^2 + p\overline{A}_t \ . \tag{C.5.8}$$

To quantize, we impose the commutation relations $[\varphi, p] = i$ (we set $\hbar = 1$ here), and we use the Schrdinger representation $p = -i\partial_\varphi$. The eigenfunctions of p and H are then the Fourier modes $\psi_n(\varphi) = \frac{1}{\sqrt{2\pi}}e^{in\varphi}$, $n \in \mathbb{Z}$. We now restrict ourselves to the case of $\theta = 2\pi k$ and $g \to \infty$, which is the case for which we are trying to calculate the electromagnetic response. Then the ground state is $\psi_k(\varphi) = \frac{1}{\sqrt{2\pi}}e^{ik\varphi} \equiv \langle\varphi|G.S.\rangle$, and the energies of all other states are pushed up to infinity (because we take the $g \to \infty$ limit). Then the partition function (vacuum-to-vacuum transition function) in this case is

$$Z[A] = \langle G.S.|e^{-iHT}|G.S.\rangle = e^{-ikT\overline{A}_t} \ , \tag{C.5.9}$$

or in terms of the original field $A = A_t dt$,

$$Z[A] = e^{-ik\int_0^T dt\ A_t} = e^{-ik\int A} \ . \tag{C.5.10}$$

The effective action is then

$$S_{eff}[A] = -i\ln(Z[A]) = -k\int A \ , \tag{C.5.11}$$

from which Eq. (C.5.2) immediately follows. Finally, we note that $Z[A]$ can also be obtained by taking the $g \to \infty$ limit of the trace tr$[e^{-iHT}]$, if we give the time a small imaginary part $T \to T - i\epsilon$ (with ϵ real and positive) such that in the $g \to \infty$ limit the trace over the entire spectrum reduces to the single term $\langle G.S.|e^{-iHT}|G.S.\rangle$. In other words, we have

$$Z[A] = \lim_{\epsilon\to 0}\lim_{g\to\infty} \text{tr}[e^{-iH(T-i\epsilon)}] \ , \tag{C.5.12}$$

where the limit $g \to \infty$ should be taken first.

Appendix D

Supplement to Chapter 5

D.1 Classical mechanics and phase space path integral for general Hamiltonian systems

In this appendix we review the symplectic geometry formulation of classical Hamiltonian mechanics, closely following the discussion in Ch. 11 of Ref. [221]. We use this formalism in Sec. 5.3 of the Chapter to aid in the evaluation of the partition function for a gauged NLSM with WZ term which describes the $(0+1)$-dimensional boundary of the BTI state in $1+1$ dimensions. The symplectic geometry formulation of Hamiltonian mechanics is a geometric formulation in terms of a phase space $\mathcal{M}$ (a closed, orientable, smooth manifold) equipped with a symplectic form ω. We take $\mathcal{M}$ to have dimension $2n$, where n is an integer greater than or equal to one. The symplectic form ω is a closed, non-degenerate two-form on $\mathcal{M}$. In a system of local coordinates m^a on $\mathcal{M}$, in which $\omega = \frac{1}{2}\omega_{ab}(\mathbf{m}) dm^a \wedge dm^b$, the non-degeneracy condition is equivalent to the condition that the components $\omega_{ab}(\mathbf{m})$ are the elements of an invertible matrix. We use the notation $\mathbf{m} = (m^1, \dots, m^{2n})$ to refer to the entire collection of phase space coordinates, and we use Latin indices near the beginning of the alphabet to label the components of general tensor fields on $\mathcal{M}$. We also use the notation $\partial_a \equiv \frac{\partial}{\partial m^a}$ and $\dot{m}^a \equiv \frac{dm^a}{dt}$ in what follows.

To start, for any function f on phase space we define an associated vector field $\underline{v}_f$ by the equation

$$df = -i_{\underline{v}_f}\omega \ , \tag{D.1.1}$$

where $i_{\underline{v}}\omega = v^a \omega_{ab} dm^b$ denotes interior multiplication of the form ω by the vector field $\underline{v}$. The components of $\underline{v}_f$ then take the form

$$v^a_f = \omega^{ab}\partial_b f \ , \tag{D.1.2}$$

where ω^{ab} are the elements of a matrix which is the inverse of the matrix with elements ω_{ab}, i.e.,

$$\omega^{ab}\omega_{bc} = \delta^a{}_c \ . \tag{D.1.3}$$

We see that the symplectic two-form ω must be non-degenerate for this to work.

The Poisson bracket of two functions f and g on phase space is then defined by[1]

$$\{f, g\} = i_{\underline{v}_g} i_{\underline{v}_f} \omega \ . \tag{D.1.4}$$

In a system of local coordinates the Poisson bracket has the form

$$\{f, g\} = \omega^{ab} \partial_b f \partial_a g \ . \tag{D.1.5}$$

For a given Hamiltonian function H, Hamilton's equations are equivalent to the single equation

$$dH = -i_{\underline{v}_H} \omega \ , \tag{D.1.6}$$

where $\underline{v}_H$ is the vector field whose components are the time derivatives of the phase space coordinates,

$$\underline{v}_H = \dot{m}^a \partial_a \ . \tag{D.1.7}$$

Finally, in each coordinate patch on $\mathcal{M}$ we can write

$$\omega = d\vartheta \ , \tag{D.1.8}$$

where the one-form $\vartheta = \vartheta_a(\mathbf{m}) dm^a$ is known as the *symplectic potential*.

Next, we review the form of the phase space path integral for the partition function $Z(T) = \text{tr}[e^{-iHT}]$ of the quantum mechanical system obtained via quantization of the classical system defined by the triple $(\mathcal{M}, \omega, H)$. Here the trace is over the Hilbert space of the quantum mechanical system. As is reviewed in Sec. 4.1 of Ref. [208], the phase space path integral for $Z(T)$ takes the form

$$Z(T) = \int_{L\mathcal{M}} [d^{2n}\mathbf{m}] \left[\prod_{t \in [0,T)} \text{Pf}[\omega_{ab}(\mathbf{m}(t))] \right] e^{iS[\mathbf{m}]} \ , \tag{D.1.9}$$

where the action appearing in the exponential is

$$S[\mathbf{m}] = \int_0^T dt \ \left[\vartheta_a(\mathbf{m}) \dot{m}^a - H(\mathbf{m}) \right] \ . \tag{D.1.10}$$

[1]The placement of $\underline{v}_f$ and $\underline{v}_g$ on the right-hand side of this equation is not a typo. We are using the non-traditional definition of the Poisson bracket from Ref. [221].

The path integral is taken over all field configurations $m^a(t)$ with periodic boundary conditions $m^a(0) = m^a(T)$ on the interval $[0,T)$. The space of all such configurations is known as the *loop space* $L\mathcal{M}$ of the phase space manifold $\mathcal{M}$. In addition, $[d^{2n}\mathbf{m}]$ denotes a flat measure on phase space at all points in time. The nontrivial geometry of the phase space is taken into account by the insertion of

$$\prod_{t\in[0,T)} \text{Pf}[\omega_{ab}(\mathbf{m}(t))] \tag{D.1.11}$$

into the path integral. This factor can be understood by noting that the $2n$-form $\frac{\omega^n}{n!}$ provides a natural volume form (the Liouville measure) on $\mathcal{M}$, and also by making use of the formula $\frac{\omega^n}{n!} = \text{Pf}[\omega_{ab}]dm^1 \wedge \cdots \wedge dm^{2n}$.

The first term in the action can also be recast into a form which is very similar to a WZ term. Let us denote the interval $[0,T)$ with periodic boundary conditions by S^1_T, the circle of circumference T. This circle is the spacetime that our quantum mechanical system evolves on. To write the first term in the action in a WZ form, we first introduce a two-dimensional manifold $\mathcal{B}$ which has S^1_T as its boundary, $\partial\mathcal{B} = S^1_T$. Then we choose an extension $\tilde{\mathbf{m}}$ of the field configuration $\mathbf{m}$ into the bulk of $\mathcal{B}$ such that $\tilde{\mathbf{m}}|_{\partial\mathcal{B}} = \mathbf{m}$. We can now use Stokes' theorem to rewrite the first term in $S[\mathbf{m}]$ as

$$\begin{aligned} \int_0^T dt\ \vartheta_a(\mathbf{m})\dot{m}^a &= \int_{S^1_T} \mathbf{m}^*\vartheta \\ &= \int_{\mathcal{B}} \tilde{\mathbf{m}}^* d\vartheta \\ &= \int_{\mathcal{B}} \tilde{\mathbf{m}}^*\omega\ . \end{aligned} \tag{D.1.12}$$

In this form the term $\int_0^T dt\ \vartheta_a(\mathbf{m})\dot{m}^a$ appearing in the action looks very similar to a WZ term, in the sense that it involves (i) an extended spacetime $\mathcal{B}$, (ii) an extension $\tilde{\mathbf{m}}$ of the field configuration $\mathbf{m}$ into $\mathcal{B}$, and (iii) the integral over $\mathcal{B}$ of the pullback of a *closed* form on $\mathcal{M}$.

D.2 A brief introduction to equivariant localization for phase space path integrals

In this appendix we give a brief review of the *equivariant localization* (EL) technique for the evaluation of certain phase space path integrals of the form of Eq. (D.1.9) from Appendix D.1. We use the EL technique in Sec. 5.3 to evaluate the partition function for a gauged NLSM with WZ term which describes the $(0+1)$-dimensional boundary of a BTI state in $1+1$ dimensions. Our presentation in this appendix is based on the discussion in Sec. 4 of Ref. [208]. We also give a brief discussion on how one can define the Pfaffians of infinite-dimensional operators which appear in the formulas obtained by applying the EL technique.

The EL technique for phase space path integrals can be thought of as an infinite-dimensional generalization of the finite-dimensional integration formulas derived in Refs. [222–224]. In this Chapter we only use the simplest version of the EL technique. The path integral formula which follows from this particular version of the EL technique is sometimes referred to as the "WKB" localization formula. This basic version of the EL method and several generalizations of it (in particular the "Niemi-Tirkkonen" formula) were developed in Refs. [204–207]. Stone's Chapter [225] on a hidden supersymmetry in the quantum mechanics of spin can be seen as a herald for the developments on the EL technique for phase space path integrals which followed soon after. The application of the EL technique to systems with a two-dimensional phase space, which is the case of interest in this Chapter, was considered in detail in Ref. [226]. Finally, some issues related to the regularization of determinants and Pfaffians appearing in the EL formulas were greatly clarified by Miettinen in Ref. [227].

In the context of the EL technique, the word "localization" refers to the fact that although the path integral in question ostensibly gets contributions from all possible field configurations, the final result only depends on contributions from a very small subset of these configurations. Thus, the integral "localizes" to a sum or, in some cases, a finite-dimensional integral over this subset of all field configurations. The word "equivariant" refers to the fact that the mechanism responsible for the localization of the integral is best understood in terms of the *equivariant cohomology* of the manifold that one is integrating over [224]. In the case of the phase space path integrals considered here this turns out to be the $U(1)$-equivariant cohomology of the infinite-dimensional *loop space* $L\mathcal{M}$ of the classical phase space $\mathcal{M}$.

The basic idea of the EL technique is as follows. First, to apply the EL technique we need to start with a classical system possessing a $U(1)$ symmetry. It turns out that this $U(1)$ symmetry "lifts" to a supersymmetry of the phase space path integral. This supersymmetry is then used to construct a one parameter family of equivalent path integrals parametrized by $\lambda \in [0, \infty)$, with the original path integral of interest corresponding to $\lambda = 0$. The supersymmetry guarantees that the path integral at any value of λ is equivalent to the original path integral. Therefore, the original path integral can be computed by taking the opposite limit $\lambda \to \infty$. In this limit the path integral simplifies dramatically, getting contributions only (in the cases considered here) from the field configurations which correspond to *time-independent* solutions to the classical equations of motion. One says that the path integral *localizes* onto these configurations. We now outline the main ideas behind the EL technique in more detail, closely following Sec. 4 of Ref. [208].

To start, we assume that it is possible to define an action of the group $U(1)$ on the phase space $\mathcal{M}$. Let $\underline{v} = v^a(\mathbf{m})\partial_a$ be the vector field which generates the $U(1)$ action, in the sense that under a $U(1)$ transformation by the small angle ξ the phase space coordinates transform as $m^a \to m^a + \xi v^a$. On $\mathcal{M}$ there is a Hamiltonian function

$H(\mathbf{m})$ which is naturally associated with this vector field, and which is determined by $\underline{v}$ and ω via the equation

$$dH = -i_{\underline{v}}\omega \ . \tag{D.2.1}$$

Note that this is just Eq. (D.1.1) with the function f taken to be the Hamiltonian. We choose this specific Hamiltonian to describe the dynamics of the system that we consider in what follows. With this choice of Hamiltonian, the action for our dynamical system will also have a $U(1)$ symmetry. Finally, we will need a Riemannian metric $g_{ab}(\mathbf{m})$ on $\mathcal{M}$ which is invariant under the $U(1)$ action generated by $\underline{v}$. This is equivalent to the requirement that $\underline{v}$ is a Killing vector for the metric, i.e., g_{ab} and v^a should satisfy the Killing equation

$$v^c\partial_c g_{ab} + g_{ac}\partial_b v^c + g_{bc}\partial_a v^c = 0, \forall\ a, b \ . \tag{D.2.2}$$

The path integral in Eq. (D.1.9) involves an integration over the loop space $L\mathcal{M}$ of $\mathcal{M}$, which is spanned by the T-periodic functions $m^a(t)$ which, for each t, represent a point on $\mathcal{M}$. We now introduce an additional set of Grassmann-valued fields $\eta^a(t)$ which also obey periodic boundary conditions. The space of these new fields is equivalent to the loop space of $\Lambda^1\mathcal{M}$, the vector space of one-forms on $\mathcal{M}$, and this space is denoted by $L\Lambda^1\mathcal{M}$. The interpretation in terms of $\Lambda^1\mathcal{M}$ is due to the fact that at each time t the anticommuting fields $\eta^a(t)$ can be regarded as a basis of one-forms on $\mathcal{M}$. Using the rules for integration over real Grassmann variables, the new fields $\eta^a(t)$ can be used to rewrite $Z(T)$ in the form

$$Z(T) = \int_{L\mathcal{M}\otimes L\Lambda^1\mathcal{M}} [d^{2n}\mathbf{m}][d^{2n}\boldsymbol{\eta}]\ e^{i(S[\mathbf{m}]+\Omega[\mathbf{m},\boldsymbol{\eta}])} \ , \tag{D.2.3}$$

where we defined

$$\Omega[\mathbf{m},\boldsymbol{\eta}] = \frac{1}{2}\int_0^T dt\ \omega_{ab}(\mathbf{m}(t))\eta^a(t)\eta^b(t) \ , \tag{D.2.4}$$

and where the integration is now over the "super loop space" $L\mathcal{M} \otimes L\Lambda^1\mathcal{M}$. One should compare Eq. (D.2.3) with the original expression Eq. (D.1.9) for $Z(T)$.

Using the Grassmann-valued fields we can define the operators

$$d_L = \int_0^T dt\ \eta^a(t)\frac{\delta}{\delta m^a(t)} \ , \tag{D.2.5}$$

and

$$i_S = \int_0^T dt\ V_S^a[\mathbf{m}(t);t]\frac{\delta}{\delta \eta^a(t)} \ , \tag{D.2.6}$$

where

$$V_S^a[\mathbf{m}(t);t] = \dot{m}^a(t) - v^a(\mathbf{m}(t)) \ . \tag{D.2.7}$$

The quantities $V_S^a[\mathbf{m}(t);t]$ can be interpreted as the components of a vector field

$$\underline{V}_S = \int_0^T dt \, V_S^a[\mathbf{m}(t);t] \frac{\delta}{\delta m^a(t)} \tag{D.2.8}$$

on the loop space. To understand the physical significance of the components $V_S^a[\mathbf{m}(t);t]$, note that the classical equations of motion for the system under consideration are

$$\frac{\delta S[\mathbf{m}]}{\delta m^a(t)} = \omega_{ab}(\mathbf{m}(t)) V_S^b[\mathbf{m}(t);t] = 0 \ , \ \forall \, a \ . \tag{D.2.9}$$

Since ω is non-degenerate, the classical equations of motion are equivalent to the equations $V_S^a[\mathbf{m}(t);t] = 0$, $\forall \, a$. The operator d_L can be interpreted as an exterior derivative on $L\mathcal{M}$, and i_S has the interpretation of interior multiplication by the loop space vector field $\underline{V}_S$.

In terms of these operators we now define the loop space equivariant exterior derivative

$$Q_S = d_L + i_S \ . \tag{D.2.10}$$

The square of this operator can be interpreted as a loop space Lie derivative (acting on loop space differential forms) along the loop space vector field $\underline{V}_S$,

$$\mathcal{L}_S \equiv Q_S^2 = d_L i_S + i_S d_L \ . \tag{D.2.11}$$

Some algebra shows that

$$Q_S(S[\mathbf{m}] + \Omega[\mathbf{m}, \boldsymbol{\eta}]) = 0 \ , \tag{D.2.12}$$

which means that the integrand in the path integral is equivariantly closed (i.e., closed under the action of the equivariant exterior derivative). To prove this relation one needs to use the fact that ω is closed as an ordinary two-form on $\mathcal{M}$, and also Eq. (D.2.9) relating $\underline{V}_S$ to the classical equations of motion.

The closure of the integrand can be interpreted in terms of a supersymmetry (SUSY) of this system which is generated by the "supercharge" Q_S. In particular, Eq. (D.2.12) implies that the path integral for $Z(T)$ is invariant under the SUSY transformation

$$\delta_\epsilon m^a(t) = \epsilon Q_S m^a(t) \tag{D.2.13a}$$

$$\delta_\epsilon \eta^a(t) = \epsilon Q_S \eta^a(t) \ , \tag{D.2.13b}$$

where ϵ is a constant Grassmann parameter. An explicit calculation gives $Q_S m^a(t) = \eta^a(t)$ and $Q_S \eta^a(t) = V_S^a[\mathbf{m}(t);t]$, so we know the exact form that this SUSY transformation takes. The next step towards establishing localization of the path integral is to use the supersymmetry to deform the path integral by adding a suitably chosen SUSY-exact term to the integrand. To this end, we modify $Z(T)$ to

$$Z(T,\lambda) = \int_{L\mathcal{M}\otimes L\Lambda^1\mathcal{M}} [d^{2n}\mathbf{m}][d^{2n}\boldsymbol{\eta}] e^{i(S[\mathbf{m}]+\Omega[\mathbf{m},\boldsymbol{\eta}])-\lambda Q_S \Psi[\mathbf{m},\boldsymbol{\eta}]} \,, \tag{D.2.14}$$

where $\Psi[\mathbf{m},\boldsymbol{\eta}]$ is some functional of $\mathbf{m}$ and $\boldsymbol{\eta}$ which will be required to satisfy

$$Q_S^2 \Psi[\mathbf{m},\boldsymbol{\eta}] = 0 \,. \tag{D.2.15}$$

If we can find such a functional $\Psi[\mathbf{m},\boldsymbol{\eta}]$, then we can show that $Z(T,\lambda)$ is actually independent of λ by the following manipulations. We compute (we suppress the arguments of the different terms for brevity)

$$\begin{aligned} \frac{dZ(T,\lambda)}{d\lambda} &= -\int [d^{2n}\mathbf{m}][d^{2n}\boldsymbol{\eta}]\; Q_S\Psi\; e^{i(S+\Omega)-\lambda Q_S \Psi} \\ &= -\int [d^{2n}\mathbf{m}][d^{2n}\boldsymbol{\eta}]\; Q_S \left[\Psi\; e^{i(S+\Omega)-\lambda Q_S \Psi}\right] \\ &= 0 \,. \end{aligned} \tag{D.2.16}$$

The second line follows from the first since the argument of the exponential is annihilated by Q_S (and this requires that $Q_S^2\Psi = 0$). Finally, the third line follows from the second due to an infinite-dimensional version of the statement that the integral of a total derivative is zero. In the infinite-dimensional case this is only true if the path integral measure is invariant under the action of Q_S, but that is the case here. An alternative explanation of the λ-independence of this integral, which uses a Ward identity associated with the symmetry generated by Q_S, can be found in Ref. [208].

The arguments from the last paragraph show that the original partition function $Z(T)$ is equal to the deformed partition function $Z(T,\lambda)$ for any value of λ. The final step in establishing the localization of $Z(T)$ is to pick a particular functional $\Psi[\mathbf{m},\boldsymbol{\eta}]$ such that the $\lambda \to \infty$ limit of $Z(T,\lambda)$ becomes easy to evaluate. There are various choices for such a $\Psi[\mathbf{m},\boldsymbol{\eta}]$, but the choice which leads to the WKB localization formula is

$$\Psi[\mathbf{m},\boldsymbol{\eta}] = \int_0^T dt\; g_{ab}(\mathbf{m}(t)) V_S^a[\mathbf{m}(t);t]\eta^b(t) \,. \tag{D.2.17}$$

One can check that this functional satisfies $Q_S^2\Psi[\mathbf{m},\boldsymbol{\eta}] = 0$, but the derivation relies on the fact that $\underline{v}$ is a Killing vector for the metric g_{ab}.

Using this particular choice of $\Psi[\mathbf{m},\boldsymbol{\eta}]$, one can now show that the path integral $Z(T)$ localizes to a sum over contributions from the field configurations in the set

$$LM_S = \{\mathbf{m}(t) \in LM \mid V^a_S[\mathbf{m}(t);t] = 0,\ \forall\, a\}\ , \tag{D.2.18}$$

which is the set of all T-periodic solutions to the classical equations of motion. To motivate this, we simply note that the bosonic term in $Q_S\Psi[\mathbf{m},\boldsymbol{\eta}]$ is

$$\int_0^T dt\ g_{ab}(\mathbf{m}(t)) V^a_S[\mathbf{m}(t);t] V^b_S[\mathbf{m}(t);t]\ . \tag{D.2.19}$$

Now $Q_S\Psi[\mathbf{m},\boldsymbol{\eta}]$ appears in the exponential of the path integral multiplied by a factor of $-\lambda$, which means that in the limit $\lambda \to \infty$, this term becomes a delta function which restricts the path integral to only those field configurations where $V^a_S[\mathbf{m}(t);t] = 0$.

The final result of the EL calculation is the formula

$$\begin{aligned} Z(T) &= \lim_{\lambda\to\infty} Z(T,\lambda) \\ &\sim \sum_{\overline{\mathbf{m}}(t)\in LM_S} \frac{e^{iS[\overline{\mathbf{m}}]}}{\text{Pf}[\mathcal{O}]_{\mathbf{m}(t)=\overline{\mathbf{m}}(t)}}\ , \end{aligned} \tag{D.2.20}$$

where the infinite-dimensional operator $\mathcal{O}$ has matrix elements

$$\begin{aligned} \mathcal{O}_{ab}(t,t') &= \frac{\delta V^c_S[\mathbf{m}(t');t']}{\delta \mathbf{m}^a(t)}\delta_{cb} \\ &= \delta_{ab}\partial_{t'}\delta(t-t') - \partial_a v^c(\mathbf{m}(t))\delta_{cb}\delta(t-t')\ , \end{aligned} \tag{D.2.21}$$

and where the notation "$\sim$" indicates equivalence up to infinite products of constant (but λ-independent) factors. The final formula Eq. (D.2.20) is famously equivalent to the stationary-phase approximation to $Z(T)$, but where the sum is taken over *all* T-periodic solutions of the classical equations of motion, and not just the solution which minimizes the action. In favorable cases there are a finite number of solutions $\overline{\mathbf{m}}(t) \in LM_S$, and the partition function reduces to a sum of finitely many terms. In addition, the Pfaffians appearing in this expression can be computed using standard regularization techniques (see, for example, Ref. [227]), as we discuss in Appendix D.3 for the examples considered in this Chapter.

We note here that there is a typo in the presentation of this formula in several original references on the EL technique. The formula presented here is the correct one and it can be found in this form in Eq. 3.13 of Ref. [205] and Eq. 13 of Ref. [227], for example. Note, however, that we present this formula in terms of an operator $\mathcal{O}$ which has all indices down, $\mathcal{O}_{ab}(t,t')$. We find that this presentation makes more sense since typically one considers the Pfaffian of an antisymmetric bilinear form $\mathcal{O}_{ab}$ and not a linear operator $\mathcal{O}^a{}_b$ which happens to be antisymmetric. In addition, in the infinite-dimensional case one needs to also properly define the Pfaffian, and with the index structure that we have chosen it is possible to define this Pfaffian in terms of a fermion path integral as we now discuss.

The Pfaffian of a $2n \times 2n$ antisymmetric matrix $\mathcal{O}_{ab}$ is a well-defined object, in the sense that there is an explicit formula for it. One way of computing the Pfaffian is by Grassmann integration. If η^a, $a = 1, \dots, 2n$, are a set of $2n$ real Grassmann variables, then we have

$$\text{Pf}[\mathcal{O}] = \int d^{2n}\boldsymbol{\eta}\, e^{-\frac{1}{2}\eta^a \mathcal{O}_{ab}\eta^b} \ , \tag{D.2.22}$$

provided that we define the measure as $d^{2n}\boldsymbol{\eta} = d\eta^1 \cdots d\eta^{2n}$. We therefore propose that in the infinite-dimensional case one should define the Pfaffian of the operator $\mathcal{O}$ via the fermionic path integral

$$\text{Pf}[\mathcal{O}] = \int [d^{2n}\boldsymbol{\eta}] e^{-\frac{1}{2}\int_0^T dt \int_0^T dt' \eta^a(t)\mathcal{O}_{ab}(t,t')\eta^b(t')} \ , \tag{D.2.23}$$

where $\eta^a(t)$ are the Grassmann-valued fields with periodic boundary conditions that we considered earlier in this section. We can then evaluate the integral by expanding the fields in Fourier modes as

$$\eta^a(t) = \sum_{m\in\mathbb{Z}} \eta^a_m \frac{e^{i\frac{2\pi m t}{T}}}{\sqrt{T}} \ , \tag{D.2.24}$$

where the Fourier coefficients η^a_m are ordinary Grassmann numbers. We also need to define the path integral measure. One possible definition is (we specialize to $n = 1$ here)

$$[d^2\boldsymbol{\eta}] = d\eta^1_0 d\eta^2_0 \prod_{m>0} d\eta^1_{-m} d\eta^1_m d\eta^2_{-m} d\eta^2_m \ , \tag{D.2.25}$$

however, the definition of the measure is ambiguous because different orderings of the terms will lead to answers which differ by an overall sign. This ambiguity is not important at this stage however, because we will eventually need to regulate the result of the path integral in order to make sense of it. We consider the careful regularization of this integral for specific examples in Sec. 5.3 of the main text and in Appendix D.3.

D.3 Evaluation of Determinants

In this appendix we compute the amplitude and phase of the regularized determinants det$[\mathcal{D}_\pm]_{reg}$ which are needed for the calculation of the partition function $Z[A]$ for the gauged boundary theory of the BTI state in Sec. 5.3 of this Chapter. We use zeta and eta functions (to be defined below) to regularize the magnitude and phase, respectively, of these determinants. The application of zeta and eta function methods to the regularization of determinants appearing in the context of EL calculations was discussed in detail by Miettinen in Ref. [227]. In particular, Miettinen showed that by defining the phase of the regularized determinant using the *eta invariant* of the operator in question, the character formula for $SU(2)$ (equivalent to the partition function for a spin in a constant magnetic field) could be obtained directly from an EL path integral calculation, without the need to correct the final answer by hand using a so-called "Weyl shift"[2].

In Sec. 5.3 we showed that the expression for the determinant of $\mathcal{D}_{\pm}$ could be manipulated into the form

$$\det[\mathcal{D}_{\pm}] = \left(\prod_{m\in\mathbb{Z}} |\lambda^{(\pm)}_m|\right) e^{i\frac{(2p+1)\pi}{2}\sum_{m\in\mathbb{Z}}(1-\text{sgn}(\lambda^{(\pm)}_m))} , \tag{D.3.1}$$

where p was an arbitrary integer. We remind the reader that $\mathcal{D}_{\pm} = -i\partial_t \pm \overline{A}_t$, and the eigenvalues of $\mathcal{D}_{\pm}$ are $\lambda^{(\pm)}_m = \frac{2\pi m}{T} \pm \overline{A}_t$, $m \in \mathbb{Z}$. In Sec. 5.3 we also showed that a regularization of the infinite sum $\sum_{m\in\mathbb{Z}} 1$ using the Riemann zeta function allowed us to reduce this expression to

$$\det[\mathcal{D}_{\pm}] = \left(\prod_{m\in\mathbb{Z}} |\lambda^{(\pm)}_m|\right) e^{-i\frac{(2p+1)\pi}{2}\sum_{m\in\mathbb{Z}}\text{sgn}(\lambda^{(\pm)}_m)} . \tag{D.3.2}$$

In this appendix we show how zeta and eta function methods can be used to carefully define the amplitude and phase in this formal expression for the determinant of $\mathcal{D}_{\pm}$.

We start with the calculation of the amplitude $\prod_{m\in\mathbb{Z}} |\lambda^{(\pm)}_m|$. To be concrete, we first assume that $\overline{A}_t \in (0, \frac{2\pi}{T})$. In this case we have

$$\prod_{m\in\mathbb{Z}} |\lambda^{(\pm)}_m| = \overline{A}_t \prod_{m>0}\left[\left(\frac{2\pi m}{T}\right)^2 - (\overline{A}_t)^2\right] . \tag{D.3.3}$$

To regularize the product on the right-hand side of this equation we first note that the ratio

$$\prod_{m>0}\left[\frac{\left(\frac{2\pi m}{T}\right)^2 - (\overline{A}_t)^2}{\left(\frac{2\pi m}{T}\right)^2}\right] = \frac{\sin\left(\frac{\overline{A}_t T}{2}\right)}{\frac{\overline{A}_t T}{2}} , \tag{D.3.4}$$

is a completely well-defined quantity. To compute this ratio we used the infinite product formula for the sine function,

$$\sin(x) = x\prod_{m=1}^{\infty}\left(1 - \frac{x^2}{\pi^2 m^2}\right) . \tag{D.3.5}$$

The product $\prod_{m>0}\left(\frac{2\pi m}{T}\right)^2$ in the denominator on the left-hand side of Eq. (D.3.4) can be interpreted as $\det'[-i\partial_t]$, where the prime indicates the determinant without the contribution from the zero mode. We can use zeta function

[2]For the spin J representation of $SU(2)$, the Weyl shift refers to the replacement of J with $J + \frac{1}{2}$ in the final answer obtained from the phase space path integral.

regularization [228] to assign a finite value to this determinant.

To apply zeta function regularization we first define a differential operator $\mathcal{P}$ with eigenvalues $\left(\frac{2\pi m}{T}\right)^2$, $m > 0$. We then define the *spectral zeta function* for this operator as

$$\zeta_{\mathcal{P}}(s) = \sum_{m>0} \left(\frac{2\pi m}{T}\right)^{-2s} , \tag{D.3.6}$$

which is well-defined for Re$[s] > \frac{1}{2}$. Then the regularized version of the determinant of $\mathcal{P}$ is defined as

$$\text{det}[\mathcal{P}]_{reg} = e^{-\zeta'_{\mathcal{P}}(0)} , \tag{D.3.7}$$

where $\zeta'_{\mathcal{P}}(0)$ is the analytic continuation of $\zeta'_{\mathcal{P}}(s)$ to $s = 0$ (and the prime denotes a derivative with respect to s). In this case the spectral zeta function $\zeta_{\mathcal{P}}(s)$ is related to the ordinary Riemann zeta function $\zeta(s)$ by

$$\zeta_{\mathcal{P}}(s) = \left(\frac{T}{2\pi}\right)^{2s} \zeta(2s) , \tag{D.3.8}$$

which means that

$$\zeta'_{\mathcal{P}}(0) = 2\ln\left(\frac{T}{2\pi}\right)\zeta(0) + 2\zeta'(0) . \tag{D.3.9}$$

Using the well-known values $\zeta(0) = -\frac{1}{2}$ and $\zeta'(0) = -\frac{1}{2}\ln(2\pi)$, we find that $\zeta'_{\mathcal{P}}(0) = -\ln(T)$, so that

$$\text{det}[\mathcal{P}]_{reg} = T . \tag{D.3.10}$$

Then, in view of the ratio Eq. (D.3.4), we define

$$\begin{aligned} \left(\prod_{m\in\mathbb{Z}} |\lambda_m^{(\pm)}|\right)_{reg} &= \overline{A}_t \, \text{det}[\mathcal{P}]_{reg} \, \frac{\sin\left(\frac{\overline{A}_t T}{2}\right)}{\frac{\overline{A}_t T}{2}} \\ &= 2\sin\left(\frac{\overline{A}_t T}{2}\right) . \end{aligned} \tag{D.3.11}$$

More generally, suppose that $\overline{A}_t$ lies in the open interval $(\frac{2\pi\ell}{T}, \frac{2\pi\ell+2\pi}{T})$ for some $\ell \in \mathbb{Z}$. In this case it is convenient to decompose $\overline{A}_t$ as

$$\overline{A}_t = \frac{2\pi\ell}{T} + \overline{a}_t , \tag{D.3.12}$$

where $\overline{a}_t \in (0, \frac{2\pi}{T})$. If we now repeat the amplitude calculation above for this case then we find that

$$\begin{aligned}\left(\prod_{m\in\mathbb{Z}} |\lambda_m^{(\pm)}|\right)_{reg} &= (-1)^\ell 2\sin\left(\frac{\overline{A}_t T}{2}\right) \\ &= (-1)^\ell 2\sin\left(\pi\ell + \frac{\overline{a}_t T}{2}\right) \\ &= 2\sin\left(\frac{\overline{a}_t T}{2}\right) \ . \end{aligned} \tag{D.3.13}$$

We now move on to the computation of the phase of the regularized determinants. First, for a complex number s with sufficiently large and positive real part, the *eta function* $\eta_\pm(s)$ of the Dirac operator $\mathcal{D}_\pm$ is defined by [71]

$$\eta_\pm(s) = \sum_{m\in\mathbb{Z}} \text{sgn}(\lambda_m^{(\pm)})|\lambda_m^{(\pm)}|^{-s} \ , \tag{D.3.14}$$

where we use the convention that sgn$(0) = 1$. This expression has a well-defined analytic continuation to $s = 0$, known as the *eta invariant*, and we use this analytic continuation to define the regularized phase of the determinant in question via the formula

$$\left(\sum_{m\in\mathbb{Z}} \text{sgn}(\lambda_m^{(\pm)})\right)_{reg} = \eta_\pm(0) \ . \tag{D.3.15}$$

We focus our attention on the calculation of the eta invariant for $\mathcal{D}_+$. The calculation for $\mathcal{D}_-$ is very similar.

First, recall that we are assuming that $\overline{A}_t$ lies in an open interval between two eigenvalues of $-i\partial_t$. This guarantees that the operators $\mathcal{D}_\pm$ do not possess any zero modes. In this case each term in $\eta_\pm(s)$ can be differentiated with respect to $\overline{A}_t$, since the value of sgn$(\lambda_m^{(\pm)})$ does not vary as we move $\overline{A}_t$ within this open interval. After taking the derivative, we find that (focusing on the case of $\mathcal{D}_+$)

$$\frac{d\eta_+(s)}{d\overline{A}_t} = -s\zeta_{\mathcal{D}_+^2}(\tfrac{s+1}{2}) \ , \tag{D.3.16}$$

where $\zeta_{\mathcal{D}_+^2}(s)$ is the spectral zeta function for $\mathcal{D}_+^2$, the square of the Dirac operator $\mathcal{D}_+$. This formula is in fact just a special case of the general formula in Proposition 2.10 of Ref. [202]. Taking the $s \to 0$ limit then gives

$$\frac{d\eta_+(0)}{d\overline{A}_t} = -\lim_{s\to 0}\ s\zeta_{\mathcal{D}_+^2}(\tfrac{s+1}{2}) \ . \tag{D.3.17}$$

The spectral zeta function $\zeta_{\mathcal{D}_+^2}(s)$ has a first order pole at $s = \frac{1}{2}$, which is due to the fact that the leading part of $\mathcal{D}_+^2$ is $-\partial_t^2$ (i.e., the dominant part of the eigenvalues of $\mathcal{D}_+^2$ for large m is the piece $\left(\frac{2\pi m}{T}\right)^2$). It then follows from Eq. (D.3.17) that $\frac{d\eta_+(0)}{d\overline{A}_t}$ is equal to *minus* the residue of $\zeta_{\mathcal{D}_+^2}(s)$ at $s = \frac{1}{2}$. This residue can be computed using the heat kernel expansion for $\mathcal{D}_+^2$, and the residue turns out to be equal to the residue of the spectral zeta function for $-\partial_t^2$

at $s = \frac{1}{2}$, which is easier to compute. From these considerations we find that

$$\frac{d\eta_+(0)}{d\overline{A}_t} = -\frac{T}{\pi} \, , \tag{D.3.18}$$

and then an integration with respect to $\overline{A}_t$ gives

$$\eta_+(0) = C_+ - \frac{\overline{A}_t T}{\pi} \, , \tag{D.3.19}$$

where C_+ is an as yet undetermined constant.

The value of the constant C_+ can be fixed uniquely by requiring the eta invariant to vanish when $\overline{A}_t$ lies halfway between two eigenvalues of $-i\partial_t$ (symmetry dictates that $\eta_+(s)$ for any s should vanish in this case). Let us assume that $\overline{A}_t \in (\frac{2\pi\ell}{T}, \frac{2\pi\ell+2\pi}{T})$ for some $\ell \in \mathbb{Z}$. Then we require $\eta_+(0)$ to vanish when $\overline{A}_t = \frac{2\pi}{T}(\ell + \frac{1}{2})$, which fixes $C_+ = 2\ell + 1$. Therefore the eta invariant is given in this case by

$$\eta_+(0) = 2\ell + 1 - \frac{\overline{A}_t T}{\pi} \, . \tag{D.3.20}$$

For the Dirac operator $\mathcal{D}_-$, and still assuming that $\overline{A}_t \in (\frac{2\pi\ell}{T}, \frac{2\pi\ell+2\pi}{T})$, all of the signs are reversed. We then find that for $\overline{A}_t \in (\frac{2\pi\ell}{T}, \frac{2\pi\ell+2\pi}{T})$, the eta invariants of the operators $\mathcal{D}_\pm$ are

$$\eta_\pm(0) = \pm(2\ell + 1) \mp \frac{\overline{A}_t T}{\pi} \, . \tag{D.3.21}$$

As in Eq. (D.3.12), it is convenient to again write $\overline{A}_t = \frac{2\pi\ell}{T} + \overline{a}_t$ with $\overline{a}_t \in (0, \frac{2\pi}{T})$. Then in terms of $\overline{a}_t$, the eta invariants for $\mathcal{D}_\pm$ take the form

$$\eta_\pm(0) = \pm 1 \mp \frac{\overline{a}_t T}{\pi} \, . \tag{D.3.22}$$

We see that the eta invariant only depends on the value of $\overline{A}_t$ modulo $\frac{2\pi}{T}$.

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